Practical Rhumb Line Calculations on the Spheroid

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1. INTRODUCTION. About ten years ago this author wrote the software for a suite of navigation programmes which was resident in a small hand-held computer. In the course of this work it became apparent that the standard text books of navigation were perpetuating a flawed method of calculating rhumb lines on the Earth considered as an oblate spheroid. On further investigation it became apparent that these incorrect methods were being used in programming a number of calculator/computers and satellite navigation receivers. Although the discrepancies were not large, it was disquieting to compare the results of the same rhumb line calculations from a number of such devices and find variations of some miles when the output was given, and therefore purported to be accurate, to a tenth of a mile in distance and/or a tenth of a minute of arc in position. The problem has been highlighted in the past and the references at the end of this paper[1-8] show that a number of methods have been proposed for the amelioration of this problem. This paper summarizes formulae that the author recommends should be used for accurate solutions. Most of these may be found in standard geodetic text books, such as Bomford[9], but also provided are new formulae and schemes of solution which are suitable for use with computers or tables. The latter also take into account situations when a near-indeterminate solution may arise. Some examples are provided in an appendix which demonstrate the methods. The data for these problems do not refer to actual terrestrial situations but have been selected for illustrative purposes only. Practising ships' navigators will find the methods described in detail in this paper to be directly applicable to their work and also they should find ready acceptance because they are similar to current practice. In none of the references cited at the end of this paper has the practical task of calculating, using either a computer or tabular techniques, been addressed.

2. RHUMB LINE PROBLEMS. The two standard problems are as follows,

(i) Problem I. Given the latitude (ϕ) and longitude (λ) of each of two points, calculate the rhumb line track (C) and the distance (S) between them.

(ii) Problem II. Given the latitude and longitude of a point and the rhumb line track and distance to a distant point, calculate the latitude and longitude of the distant point.

The geometrical relationship between the quantities is seen in Figures 1 and 2, where $\Delta\lambda$ is the difference in longitude between X and Y, ΔM is the difference in meridional parts and Δm is the difference in meridional distances. It should be

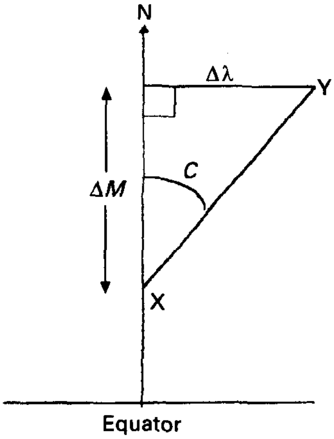

Fig. 1. Chart

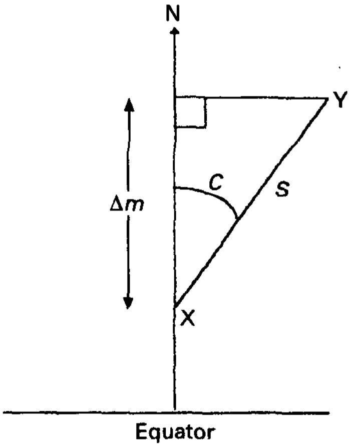

Fig. 2. Spheroid

noted that the distance along the meridian between the latitudes of X and Y is not $\Delta\phi$, the difference in latitude, but the difference in the lengths of the meridian arcs (meridian distances) from the equator. Use of $\Delta\phi$ has been the approximation which, in the past, has given rise to the discrepancies referred to in the introduction.

3. FORMULAE. The following are appropriate for the solution of problems I and II.

$$M = \frac{10800}{\pi}\left(\ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-\frac{1}{2}e\ln\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)\right) \qquad (1)$$

where e is the eccentricity of the spheroid.

$$m = \frac{a}{1852}(A_0\,\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi\ldots\ldots) \qquad (2)$$

where
$$A_0 = 1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256} \dots\dots\dots$$
$$A_2 = \frac{3}{8}\left(e^2 + \frac{e^4}{4} + \frac{15e^6}{128} \dots\dots\dots\right)$$
$$A_4 = \frac{15}{256}\left(e^4 + \frac{3e^6}{4} \dots\dots\dots\dots\right)$$
$$A_6 = \frac{35e^6}{3072} \dots\dots\dots\dots\dots\dots$$

where a is the semi-major axis of the spheroid and m is in international nautical miles.

$$\phi = \frac{1852m}{aA_0} + \frac{A_2}{A_0}\sin 2T + \frac{7A_4}{5A_0}\sin 4T + \frac{11A_6}{5A_0}\sin 6T \dots\dots\dots \qquad (3)$$

where T is the first term on the right hand side of the expression.

$$S = \frac{a\Delta\lambda\cos\bar{\phi}}{1852\sin C(1 - e^2\sin^2\bar{\phi})^{\frac{1}{2}}} \qquad (4)$$

where $\bar{\phi}$ is the mean latitude and S is in nautical miles.

$$\Delta\lambda = \frac{1852\sin C(1 - e^2\sin^2\bar{\phi})^{\frac{1}{2}}}{a\cos\bar{\phi}} \qquad (5)$$

$$S = \frac{\Delta\lambda}{\sin C}(\cos\bar{\phi} + P) \qquad (6)$$

$$\Delta\lambda = S\sin C\left(\frac{1}{\cos\bar{\phi}} - Q\right) \qquad (7)$$

P and Q will be defined and explained in the following section. There are more than sufficient terms in formulae (2) and (3) for most marine applications. The user may therefore choose the number of terms that are sufficient for the application under consideration.

4. PRACTICAL CALCULATIONS.

4.1. *Computer methods.*

Problem I.

Calculate M_X, M_Y, m_X and m_Y using formulae (1) and (2). Solve for C and S from $\tan C = \Delta\lambda/\Delta M$ and $S = \Delta m/\cos C$, where $\Delta\lambda = \lambda_Y - \lambda_X$, $\Delta M = M_Y - M_X$ and $\Delta m = m_Y - m_X$.

Heed the ratio of signs in the expression for tan C to place C in its correct quadrant. The units of M have been chosen to match those of $\Delta\lambda$ (see Appendix). If $\phi_X = \phi_Y$, that is $C = 90°$ or $270°$, and also if the track lies close to the east or west directions, use formula (4) to find S because, in the latter case, the distance S, derived from $S = \Delta m/\cos C$, may be inaccurate as both Δm and cos C are very small. This will be especially so if the computing device works to a small number of significant digits.

Problem II.

Calculate m_X using formula (2), $\Delta m = S \cos C$ and thence find $m_Y = m_X + \Delta m$.

ϕ_Y may then be obtained from formula (3) which has been derived empirically by reversing the series formula (2). However, it is not essential to have formula (3) in a computer program because latitude can be derived by successive approximations using an initial value of $\phi = m/60$ etc. in formula (2). Three iterations are sufficient for most marine navigation applications.

For example, given $m = 2400$, find ϕ.

Iteration	m	Δm	$\Delta\phi$	ϕ
Start	2400			40°
1	2391·7542	8·2458	0·13743	40·13743
2	2399·9938	0·0062	0·00010	40·13753
3	2399·9998	....	....	....

Calculate M_X, M_Y and $\Delta\lambda = \Delta M \tan C$.

As in Problem I, for a course lying very close to a parallel, $\Delta\lambda$ derived from $\Delta\lambda = \Delta M \tan C$ may be inaccurate because ΔM is very small and $\tan C$ very large. Under these circumstances formula (5) should be used. For a course along the parallel, formula (5) is exact.

4.2. *Tabular methods.* It will be assumed that the user possesses a table of spheroidal meridional parts (M) such as those which are to be found in *Norie* and *Bowditch*. It is not essential, although it is preferable, to use the parameters of a recently adopted spheroid. For example, comparing WGS 72 and WGS 84, the difference between the semi-major axes is two metres and 1×10^{-5} in the flattening. For the methods described here, in addition to a table of meridional parts, a table of meridional distances and P and Q will be necessary. At an argument interval of one degree the table is quite small as can be inferred from the extracts which are provided for the illustrative examples.

Problem I.

From the tables find M_X, M_Y, m_X, and m_Y. Using a calculator solve C from $\tan C = \Delta\lambda/\Delta M$ and S from $S = \Delta m/\cos C$.

The problem of an unstable solution for S for lines in the vicinity of the east and west directions is worse than in the computer solution because of the severe loss of significant digits, particularly with Δm. For a sphere a very accurate approximation to the distance along near east or west lines, see reference (5), is:

$$S = \frac{\Delta\lambda}{\sin C} \cos \bar{\phi}$$

By analogy the spheroidal counterpart of this formula would be formula (4) which may be put in the form of formula (6):

where $$P = \cos\bar{\phi}\left(\frac{a}{1852\,(1-e^2\sin^2\bar{\phi})^{\frac{1}{2}}} - 1\right)$$

or $$P = \cos\bar{\phi}\left(\frac{a\pi}{60\times 1852\times 180\,(1-e^2\sin^2\bar{\phi})^{\frac{1}{2}}} - 1\right)$$

if S is expressed in nautical miles and $\Delta\lambda$ in tenths of minutes of arc. P, in the range of latitude normally encountered in marine navigation, is a small, almost linear, quantity when tabulated at degree intervals to 0·00001.

Problem II.

From the tables find m_X. Using a calculator solve for Δm from $\Delta m = S\cos C$ and thence m_Y from $m_Y = m_X + \Delta m$. ϕ_Y may be found from the tables by interpolating m_Y through the respondent. From the tables find M_X and M_Y, $\Delta\lambda$ from $\Delta\lambda = \Delta M\tan C$ and thence λ_Y.

The inaccurate solution of $\Delta\lambda$ using $\Delta\lambda = \Delta M\tan C$ in the vicinity of a parallel may be avoided in a similar way to that used in Problem I.

For a sphere:

$$\Delta\lambda = \frac{S\sin C}{\cos\bar{\phi}}$$

and the corresponding expression for a spheroid is formula (5), which may be put in the form of formula (7):

where $$Q = \frac{1}{\cos\bar{\phi}}\left(1 - \frac{1852(1-e^2\sin^2\bar{\phi})^{\frac{1}{2}}}{a}\right)$$

or $$Q = \frac{1}{\cos\bar{\phi}}\left(1 - \frac{60\times 1852\times 180(1-e^2\sin^2\bar{\phi})^{\frac{1}{2}}}{a\pi}\right)$$

when adopting the same practical units as in Problem I.

Examples of the solution of the two standard problems, including the critical case of nearly east-west courses, are contained in the appendix. Unlike the methods proposed by most other authors, which give results in geographical miles, these formulae give distances in international nautical miles of 1852 metres.

5. CONCLUSION. It has been shown that the precise calculation of rhumb line problems, either using a computer or tables and a calculator, can be made in a straightforward manner. It is only when the rhumb lines lie on or very close to a parallel of latitude that special formulae need be used. For tabular methods of solution the user needs, in addition to a table of meridional parts, which may be found in standard navigation tables such as *Norie* and *Bowditch*, a small table (about half a page) of meridional distances (m) and the factors P and Q. A table of m, P and Q, extracts from which have been given for the examples, has been constructed at degree intervals and has proved to be adequate in the range of latitudes normally encountered in marine navigation. The additional work required to effect an accurate solution is slight as will have been seen from the

examples given. The material contained in this article was transmitted to H.M. Nautical Almanac Office in July 1993 and has been of assistance in the preparation of a computer navigational program there.

APPENDIX: EXAMPLES

Extracts from Tables of M, m, P, and Q (WGS 84)

Lat	M	Lat	m	Lat	P	Lat	Q
4° 40′	278·44	4°	238·83	48°	0·00244	11°	0·00195
4° 41′	279·43	5°	298·53	49°	0·00243	12°	0·00198
22° 11′	1356·86	22°	1314·15	52°	0·00239	24°	0·00257
22° 12′	1357·94	23°	1373·94	53°	0·00237	25°	0·00264
		24°	1433·74				
52° 47′	3723·86	25°	1493·55				
52° 48′	3725·51						
53° 10′	3761·96						
53° 11′	3763·63						

Example 1. Find the course and distance from 10° 18.′4N 037° 41′.7E *to* 53° 29.′5N 113° 17′.1E.

	Latitude		Longitude		M		m
Finish	53° 29.′5N		113° 17′.1E		3794·54		3201·59
Start	10° 18.′4N		037° 41′.7E		617·64		615·43
		$\Delta\lambda$	75° 35′.4E	ΔM	3176·90	Δm	2586·16

$$\tan course = \frac{\Delta\lambda}{\Delta M} = \frac{4535{\cdot}4}{3176{\cdot}90} \qquad Course\ 54°.9900$$

$$\text{Distance} = \frac{\Delta m}{\cos course} = \frac{2586{\cdot}16}{\cos 54°{\cdot}9900} \qquad 4507{\cdot}7$$

Course 055°.0 Distance 4507·7 n.m.

Example 2. Given the starting position, course and distance, find the final position. Start 22° 11′.4N 115° 44′.2W; course 237°.6; distance 2994 n.m.

m for start (22° 11′.4N)	1325·51	
Δm = *Distance* cos *course* = 2994 *cos* 237°.6	−1604·27	
m for finish	−278·76	
Latitude of finish		04° 40′.1S
M for finish (04° 40′.1S)	−278·54	
M for start (22° 11′.4N)	1357·29	
ΔM	−1635·83	
Longitude of start		115° 44′.2W
$\Delta\lambda = \Delta M$ tan *course* = −1635·83 tan 237°.6	−2577·7	42° 57′.7W
Longitude of finish		158° 41′.9W

Latitude 04° 40′.1S Longitude 158° 41′.9W

Example 3. Find the course and distance from 52° 47.′8S 097° 31′.6W to 53° 10.′8S 041° 34′.6W.

	Latitude		Longitude		M
Finish	53° 10.′8S		041° 34′.6W		−3763.30
Start	52° 47.′8S		097° 31′.6W		−3725·18
Mean latitude	52° 59.′3S	$\Delta\lambda$	55° 57′.0E	ΔM	−38·12
P	0·00237				

$$\tan course = \frac{\Delta\lambda}{\Delta M} = \frac{3357{\cdot}0}{-38{\cdot}12} \qquad \text{Course } 90°.6506$$

$$\text{Distance} = \frac{\Delta\lambda}{\sin course}(\cos MeanLatitude + P)$$

$$Distance = \frac{3357{\cdot}0}{\sin 90°.6506}(\cos -52°\ 59'.3 + 0{\cdot}00237) \qquad 2028{\cdot}9$$

Course 090°.7 Distance 2028·9 n.m.

Example 4. Find the course and distance from 48° 45.′0N 061° 31′.1W
to 48° 45.′0N 005° 13′.2E
$\Delta\lambda$ 66° 44′.3E

Mean latitude	48° 45.′0N
P	0·00243
$\Delta\lambda$	4004′.3
Course	090°

$$\text{Distance} = \frac{\Delta\lambda}{\sin course}(\cos MeanLatitude + P)$$

$$\text{Distance} = \frac{4004{\cdot}3}{\sin 90°}(\cos 48°\ 45'.0 + 0{\cdot}00243) \qquad 2649{\cdot}9$$

Course 090° Distance 2649·9 n.m.

Example 5. Given the starting position, course and distance, find the final position. Start 23° 44.′7N 045° 22′.2W; course 271°.1; distance 3508 n.m.

m for start (23° 44.′7N)	1418·49	
$\Delta m = Distance \cos course = 3508 \cos 271°.1$	67·34	
m for finish	1485·83	
Latitude of finish		24° 52.′3N

Mean Latitude 24° 18.′5N

Q 0·00259

$$\Delta\lambda = Distance \sin course\left(\frac{1}{\cos MeanLatitude} - Q\right)$$

$$\Delta\lambda = 3508 \sin 271°.1\left(\frac{1}{\cos 24°\ 18'.5} - 0{\cdot}00259\right) - 3839'.5$$

Longitude of start		045° 22′.2W
$\Delta\lambda$	−3839′.5	63° 59′.5W
Longitude of finish		109° 21′.7W

Latitude 24° 52.′3N Longitude 109° 21′.7W

Example 6. Given the starting position, course and distance, find the final position. Start 11° 13.′2S 103° 12′.3E; course 270°; distance 2536 n.m.

Mean latitude 11° 13.′2S
Q 0·00196

$$\Delta\lambda = Distance \sin course \left(\frac{1}{\cos MeanLatitude} - Q\right)$$

$$\Delta\lambda = 2536 \sin 270° \left(\frac{1}{\cos - 11°\ 13'.2} - 0{\cdot}00196\right) \qquad -2580'.4$$

Longitude of start		103° 12′.3E
$\Delta\lambda$	−2580′.4	43° 00′.4W
Longitude of finish		060° 11′.9E

Latitude 11° 13.′2S Longitude 060° 11′.9E

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KEY WORDS

1. Rhumb lines. 2. Loxodromes. 3. Geodesy.