Martelli's tables of 1920

From the well-known altitude formula

$$\sin h = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t$$

where h is the altitude, φ is the latitude, δ is the declination and t is the local hour angle respectively, we can solve for $\cos t$

$$\cos t = \frac{\sin h - \sin\varphi\sin\delta}{\cos\varphi\cos\delta}$$

By subtracting each side from unity, we get

$$1 - \cos t = 1 - \frac{\sin h - \sin\varphi\sin\delta}{\cos\varphi\cos\delta}$$

and further

$$1 - \cos t = \frac{\cos\varphi\cos\delta - \sin h + \sin\varphi\sin\delta}{\cos\varphi\cos\delta}$$

which equals

$$1 - \cos t = \frac{\cos(\varphi - \delta) - \sin h}{\cos\varphi\cos\delta}$$

Inverting, we have

$$\frac{1}{1 - \cos t} = \frac{\cos\varphi\cos\delta}{\cos(\varphi - \delta) - \sin h}$$

This is the formula used by Martelli's tables, expressed in sines and cosines only (other forms including versines and/or haversines can also be derived).

If we want to solve directly for *t*, then we have $t = \cos^{-1}\left(1 - \frac{\cos(\varphi-\delta) - \sin h}{\cos\varphi\cos\delta}\right)$ (1)

Solving the Martelli formula by logarithms, we get

$$\log\frac{1}{1-\cos t} = \log\cos\varphi + \log\cos\delta + \log\frac{1}{\cos(\varphi-\delta) - \sin h}$$

In this expression, the first two terms on the right hand side are in principle given by table I (log of lat and declination) and the third term by table IV (auxiliary logarithm). But to enter table IV we have to use table II (sum or difference) to calculate the first term in the denominator, and table III (angle of altitude) to find the second term in the denominator. The left hand side is given by table V (log of hour angle).

But the tables do contain certain constants, presumably to avoid negative numbers, and certain multipliers in order to avoid decimals.

Table I gives

$10^4\,(0.5 + \log\cos x)$ where *x* is either φ or δ and the logarithm is to the base of 10. The choice of constant 0.5 limits the argument *x* to 71°34' for non-negative table results.

Table II gives

$10^3\,(0.2 + \cos(\varphi - \delta))$ with the result in seconds (of time, according to Martelli's examples), but converted to minutes and seconds. The largest possible value of $\cos(\varphi - \delta)$ is unity, therefore the table starts with 1200^s, or 20^m.

Table III gives

$10^3(1 - \sin h)$ also with the result (in seconds) converted to minutes and seconds.

Table IV gives

$10^4\left(\log\frac{21600}{x-1200} - 1\right)$ where *x* is the sum of the results obtained by table II and table III respectively, expressed in seconds.

Finally, **table V** gives

$\log\frac{21.6}{1-\cos t}$ with *t* expressed in time, where 1 hour of time corresponds to an arc of 15°.

Note that the sum of table I (used twice) and table IV has to be divided by 10^4 to get the decimals corresponding to table V.

Now let us see if and how these expressions make sense. We start with the difference between latitude and declination and enter table II. Summed with table III we obtain

$$10^3\,(0.2 + \cos(\varphi - \delta)) + 10^3(1 - \sin h)$$

which is used as an argument for table IV. Out from table IV we thus get

$10^4\left(\log\frac{21600}{10^3\,(0.2+\cos(\varphi-\delta))+10^3(1-\sin h)-1200} - 1\right)$ which simplified equals $10^4\left(\log\frac{21.6}{\cos(\varphi-\delta)-\sin h} - 1\right)$

This expression has to be added to the outputs from table I and table II respectively. Thus we have

$$10^4\,(0.5 + \log\cos\varphi\,) + 10^4\,(0.5 + \log\cos\delta\,) + 10^4\left(\log\frac{21.6}{\cos(\varphi-\delta) - \sin h} - 1\right)$$

that is simplified to

$10^4\left(\log\cos\varphi + \log\cos\delta + \log\frac{21.6}{\cos(\varphi-\delta)-\sin h}\right)$ or

$$10^4\log\frac{21.6\ \cos\varphi\ \cos\delta}{\cos(\varphi-\delta) - \sin h}$$

Now, as mentioned above, this expression has to be divided by 10^4 to get the decimals correct before entering table V. We thus have

$\log\frac{21.6\ \cos\varphi\ \cos\delta}{\cos(\varphi-\delta)-\sin h}$ as input to table V. This table is used backwards, i.e. we look for this logarithm and then find the hour angle as the argument to the table. Used backwards table V thus delivers

$t = \cos^{-1}\left(1 - \frac{21.6}{10^{\log\frac{21.6\ \cos\varphi\ \cos\delta}{\cos(\varphi-\delta)-\sin h}}}\right)$ which is equal to $t = \cos^{-1}\left(1 - \frac{\cos(\varphi-\delta)-\sin h}{\cos\varphi\ \cos\delta}\right)$, corresponding to equation (1) above.