Solving the Nautical Triangle by Hand

I was challenged to do CelNav arithmetic by hand just as Capt J Cook did. So I started out looking for the most simple arithmetic method for evaluating hc by hand and went through my books. Well, in a Russian math compendium, translated to English, I found a trig formula I'd never seen before (ISBN 978-3-540-72121-5 / pg 81 / equ 2.128). It has features that are rather unusual but desirable for the navigator.

Look at this: We are of course used to

$$\sin(hc) = \sin(D)*\cos(L) + \cos(D)*\sin(L)*\cos(t).$$

The book shows that this expression can be transformed, after some algebra, into:

$$\sin(hc) = [\cos(D-L) - \cos(D+L)] / 2 + [\cos(S-2D) + \cos(S-2L) + \cos(S-2t) + \cos(S)] / 4$$

with the definition $\mathbf{S = D + L + t.}$

There are methods using auxiliary trig functions for multiplication that have been used even longer than logarithms. They were known by the name of *prothaphaeresis.* Don't ask me how to pronounce this - I will call it the *p - method*. The original expression asks for 3 multiplications and 1 addition. As you know multi-digit multiplications as required in this expression are tough and error prone. The p – method, however, transforms the original expression into a series of additions/subtractions of six cos() modified by trivial divisions by 2 and 4.

It is true, logarithms allow to execute multiplications by additions as well but they have their own significant problems. In particular, they cannot deal with negative numbers at all, also not with zero.
The p-method, on the other hand, is much simpler, more accurate and does not exhibit such problems.

When you google *prothaphaeresis* you will get the impression that the p - method works only for a multiplication of 2 numbers. But as the book shows, it can be extended to the multiplication of at least 3 numbers. For fun let's try it with a calculator using simple numbers: D=10°, L=20°, t= 30°.

It is: $\sin(hc) = \sin(10^\circ) * \sin(20^\circ) + \cos(10^\circ) * \cos(20^\circ) * \cos(30^\circ) = \mathbf{0.860\ 825\ 441}$

Now, the p-method: $S = 10^\circ + 20^\circ + 30^\circ = 60^\circ$;

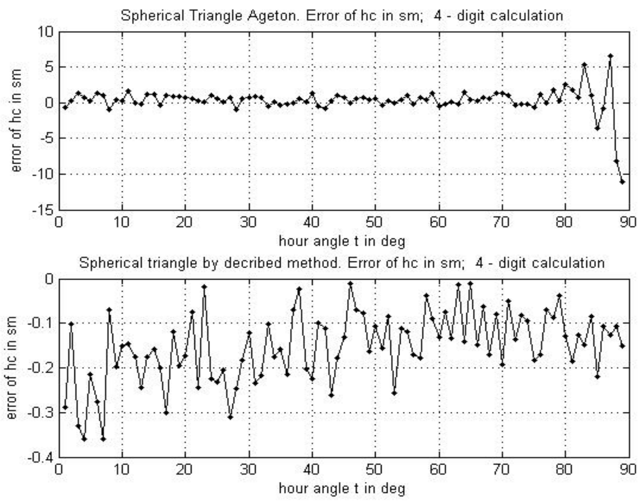
$$\sin(hc) = [\cos(10^\circ - 20^\circ) - \cos(10^\circ + 20^\circ)] /2 + [\cos(60^\circ - 2*10^\circ) + \cos(60^\circ - 2*20^\circ) + \cos(60^\circ - 2*30^\circ) + \cos(60^\circ)] /4$$

$$= [\cos(-10^\circ) - \cos(30^\circ)]/2 + [\cos(40^\circ) + \cos(20^\circ) + \cos(0^\circ) + \cos(60^\circ)] / 4 =$$

= **0.860 825 441 :)** BTW: These calculations were done with MATLAB.

I also wrote MATLAB simulations implementing the p - method and Ageton's method using a variable number of digits for the calculations. (Dreisonstok's results are practically identical to Ageton's). As a reference I used the original equation for hc as quoted in the first paragraph above.

Surprisingly, even with only 4 digits for the cos() the p - method will have an error < ± 0.5 arc min according to the numerous scenarios I executed. Ageton's and Dreisonstok's exhibited normally errors of ± 1 arc min for hour angles up to 80 degrees. For larger hour angles, however, their errors jump up irregularly but drastically to reach 10 arc min above 85 degrees. In contrast, the p – method's error stayed within the original error window. See some results in the fig below. Granted, making numerous examples does not mathematically proof anything. However, the fact that I never saw errors > 1 arc min certainly boosted my confidence.



D =45° 52'
L =20° 24'
t =18° 9'

Observe the big difference of the two vertical axis!

In another vein: When doing sight reduction by hand I regularly stumble when working on sexagesimal calculations, as for instance: *(21º, 45', 3.25”) - (12º, 53', 45.68” sec)*. So I designed the worksheet for the p-method using exclusively multiples of 1 arc-min or, equivalently, sea miles. Sea miles were what J. Cook was actually interested in. He didn't expect to exceed 1 sm of resolution/accuracy, and virtually all angles he dealt with arithmetically reached from 0º to 90º i.e. from 0' to 5400'. Numbers from 0 to 5400 are expressed in 4 digit integers which I judge sufficient for the p - method. With this change we are doing our arithmetic solely with integers and entirely in the decimal system. This could – and should! - be done in CelNav in general.

Consider this: In the Royal Navy, even in the early 20th century, calculations were executed with the inclusion of arc secs which necessitated at least six digit log calculations. See J. Huntington, “Navigation and Nautical Astronomy”. I believe the intent was to achieve the highest possible accuracy under the working conditions at sea. But this was done without a conclusive analysis whether it was actually achieved or not. I personally doubt it was. The 11th edition of Huntington's book itself has several significant arithmetic errors, such as in his first example in paragraph 281. I am amazed they were never corrected after so many editions. I assume if this authority made repeatedly significant errors which went undetected for a long time then shipborne navigators on duty did so too – and probably frequently as well. And those poor navigators had no cross check, no real independent method to solve the same problem, no redundancy. The result must have been errors and a huge waste of effort and time!

But then again, they could have evaluated the formula directly for a check even with some reduction of the digits. Even I, a terrible reckoner, found it not too bad when restricting myself to 4 digits which I found quite OK. I recommend that for the p – method as well, it will give you hints about the numbers, signs, etc.

In summary, I believe the p - method method offers an alternative with superior accuracy for doing sight reduction by hand. Notice that it needs no special slide rules, nor a variety of special trig functions, nor particular considerations of critical cases and no rules other than the classic algebraic sign and trig rules. Also, it is easy to teach, to memorize and to apply. Therefore, it enriches the toolbox of the navigator with a surprisingly simple tool which is useful when the batteries are empty. As a minimum, it provides an independent cross check to the classic log method.

For your convenience I have designed a worksheet and a cos() table for angles in arc min. To compress the table to 2 pages the angles proceed by 2 arc min in general which makes interpolation a snap. You will find all this and 2 examples in the attachments. Consider this material drafts.

Note: Given the age of the field I assume somebody else has thought of the p - method long ago - I am not likely the original inventor of it. Anyway, I'd be happy if fellow CelNav aficionados found it useful on occasion.

I am eager to hear from you! H.D.