we replace z, p, b, and $\measuredangle ZPO$ with $90° - h$, $90° + \delta$, $90° - \varphi$, and $180° - T$ (T being understood to represent the time angle of the sun), and we obtain

$$-\cos T = \tan \delta \tan \varphi + \frac{\sin h}{\cos \delta \cos \varphi}.$$

This yields the true local time T of the A.M. observation

$$\text{T.L.T.} = T = 134° 47.5' = 8 \text{ hr } 59 \text{ min } 10 \text{ sec.}$$

From this and the time equation e we obtain the mean local time of the observation

$$\text{M.L.T.} = \text{T.L.T.} + e = 8 \text{ hr } 44 \text{ min } 7 \text{ sec.}$$

If we reduce the mean Greenwich time of the observation by the mean local time, we obtain the western longitude λ of the observation point in time:

$$\lambda = \text{M.G.T.} - \text{M.L.T.} = 10 \text{ hr } 5 \text{ min } 53 \text{ sec.}$$

In angular measure (1 hr time longitude = 15 degrees longitude), this comes to

$$\lambda = 151° 28.25' \text{ W.}$$

IV. Determination of the meridian longitude Λ.

$$\Lambda = \lambda + l = 151° 48'.$$

Result: A.M. Position: 44° 38.5′ N, 151° 28.25′ W,
Noon Position: 44° 44.3′ N, 151° 48′ W.

79 Gauss' Two-Altitude Problem

From the altitudes of two known stars determine the time and position.

This problem, which is very important for astronomers, geographers, and mariners, was solved by Gauss in 1812 in Bode's *Astronomisches Jahrbuch.*

Two stars are said to be known when their equatorial coordinates—the right ascension and declination—are known. Let these coordinates of the two stars S and S' be $\alpha|\delta$ and $\alpha'|\delta'$. In the present problem all we need in addition is the right ascension difference $\alpha' - \alpha$. In the figure let P be the world pole; thus $PS = p = 90° - \delta$

will be the pole distance from S; $PS' = p' = 90^\circ - \delta'$ will be the pole distance from S'; and $\measuredangle SPS' = \tau$ will be the angle between the hour circles of the two stars, as well as the magnitude of the right ascension difference; let Z be the zenith of the observation point, so that $PZ = b = 90^\circ - \varphi$ is the complement of the latitude φ, $ZS = z$ the zenith distance from S, and $ZS' = z'$ the zenith distance from S', the last two being as well the complements of the altitudes h and h', respectively.

We still need the auxiliary magnitudes $\measuredangle PSS' = \sigma$, $\measuredangle PS'S = \sigma'$, $\measuredangle PSZ = \psi$, $\measuredangle ZSS' = \zeta$, $\measuredangle ZPS = t$, and the side $SS' = s$.

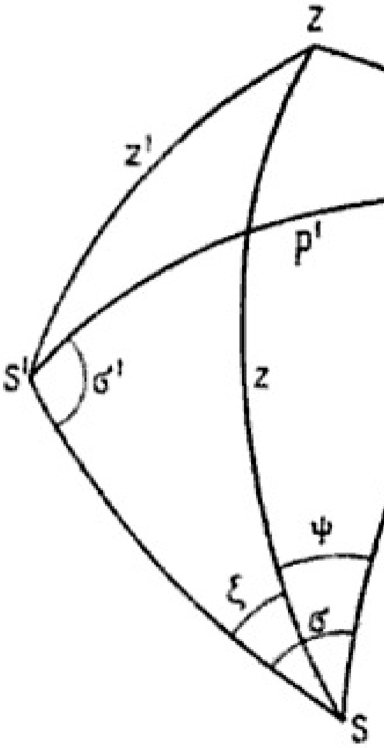

Fig. 94.

The computation, which is very simple, consists of three steps corresponding to the three triangles PSS', ZSS', PZS, which are taken up in that order.

I. Triangle PSS'. The angles σ and σ' are determined according to Napier's formulas

$$\tan\frac{\sigma+\sigma'}{2} = \frac{\cos\dfrac{p'-p}{2}}{\cos\dfrac{p'+p}{2}}\cot\frac{\tau}{2} \qquad \Bigg| \qquad \tan\frac{\sigma-\sigma'}{2} = \frac{\sin\dfrac{p'-p}{2}}{\sin\dfrac{p'+p}{2}}\cot\frac{\tau}{2}$$

and the side s is determined according to the sine formula

$$\sin s : \sin p = \sin\tau : \sin\sigma'.$$

DOUWES'* PROBLEM: *From two altitudes of a star (the sun) with known declination and the interval between the two observations determine the latitude of the place of observation.*

We need only consider S and S', respectively, as the place, δ and δ', respectively, as the declination of the star at the first and second observations. For fixed stars $\delta = \delta'$, while for the sun and the planets δ' differs somewhat from δ. (τ is the angle determined by the known time interval between the hour circles of the star corresponding to the two moments of observation.)

Since the two measured altitudes are usually observed at *different* places A and B, while the above calculation is related to only *one* place, let us say B, the altitude measured at A must be "reduced to place B." For this purpose we solve the problem:

At a place A *the altitude of a star is observed at a given time* $\mathfrak{Z}$*; at the same moment in time what is the altitude of the star at place* B*?*

To begin with, it is clear that all places on the earth's surface at which the star has the same altitude or the same zenith distance at moment $\mathfrak{Z}$ lie on a *circle* of the geosphere the spherical midpoint of which is the end point S_0 of the earth radius from the geocenter to the star. This circle is called the *equal altitude circle* of the star, its midpoint S_0 the *star image*.

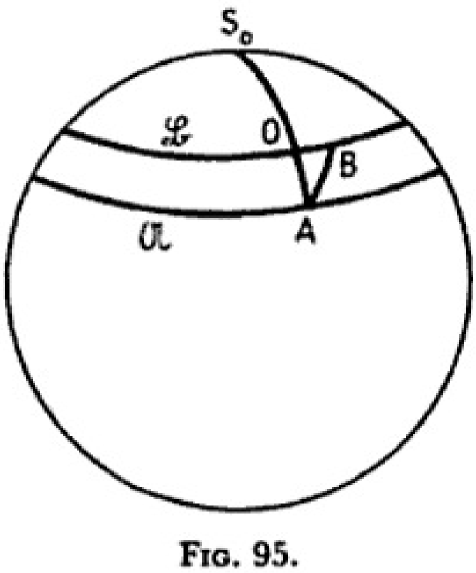

FIG. 95.

In Figure 95 let $\mathfrak{A}$ and $\mathfrak{B}$ be the two equal altitude circles of the star at moment $\mathfrak{Z}$ on which the observation points A and B lie; let S_0 be the star image, O the point of intersection of the great arc S_0A with $\mathfrak{B}$. We will assume that the distance AB is so small that the triangle AOB can be considered plane. This gives for the difference between

* Douwes was a Dutch admiralty mathematician.

the zenith distances and, consequently, also for the difference in the altitudes of the star at A and B

$$AO = AB \cos \omega,$$

where ω is the angle between the ship's course AB and the bearing AO of the star at A.

We accordingly obtain the sought-for star altitude h at B at the time $\mathfrak{Z}$ of the observation made at A if we increase or reduce the star altitude measured at A by the product of the traversed distance AB and the cosine of the angle between the course and the bearing of the star at A, accordingly as the ship draws nearer to or recedes from the star.

The "reduced" altitude thus obtained must then be substituted for h in the above Gauss equation, while the altitude measured at B must be used for h'.

The value for φ obtained by this calculation is naturally the latitude of the second observation point B.

80 Gauss' Three-Altitude Problem

From the time intervals between the moments at which three known stars attain the same altitude, determine the moments of the observations, the latitude of the observation point and the altitude of the stars.

The significance of this Gauss method for determining time and location resides in the fact that it eliminates all observational error resulting from atmospheric refraction.

Solution. We designate the equatorial coordinates (right ascension and declination) of the three stars as $\alpha|\delta$, $\alpha'|\delta'$, $\alpha''|\delta''$, the latitude of the observation point as φ, the moments of the observations as t, t', t'', the time angles of the three stars at these moments as T, T', T'', so that the differences $T' - T = t' - t$ and $T'' - T = t'' - t$ are known. This gives us the three equations

(1) $$\sin h = \sin \delta \sin \varphi - \cos \delta \cos \varphi \cos T,$$

(2) $$\sin h = \sin \delta' \sin \varphi - \cos \delta' \cos \varphi \cos T',$$

(3) $$\sin h = \sin \delta'' \sin \varphi - \cos \delta'' \cos \varphi \cos T''.$$

By subtracting the two first equations we obtain

(4) $$\sin \varphi(\sin \delta - \sin \delta') = \cos \varphi(\cos \delta \cos T - \cos \delta' \cos T').$$

We now introduce the half sum and half difference

$$s = \frac{\delta' + \delta}{2} \quad \text{and} \quad u = \frac{\delta' - \delta}{2}$$

and

$$S = \frac{T' + T}{2} \quad \text{and} \quad U = \frac{T' - T}{2}$$

of the declinations δ' and δ and the time angles T' and T, respectively, and accordingly replace δ' and δ in (4) by $s + u$ and $s - u$, and replace T' and T by $S + U$ and $S - U$. In the transformed equation (4) we then apply the addition theorem throughout and obtain

$$-\sin\varphi\cos s\sin u = \cos\varphi(\sin S\sin U\cos s\cos u + \cos S\cos U\sin s\sin u).$$

Here we divide by $\cos\varphi\cos s\sin u$ and obtain

$$-\tan\varphi = \sin S\cdot\sin U\cot u + \cos S\cdot\cos U\tan s.$$

Since U, u, and s are known, we determine the auxiliary magnitudes r and w such that

$$r\cos w = \sin U\cot u \quad \text{and} \quad r\sin w = \cos U\tan s.$$

(First w is determined from $\tan w = \tan s\tan u\cot U$ and then r from one of the two auxiliary equations.) The equation obtained then assumes the simple form

$$\text{(I)} \qquad -\tan\varphi = r\sin[S + w].$$

In precisely the same way, by subtracting the two equations (1) and (3), introducing the half sums

$$\mathfrak{s} = \frac{\delta'' + \delta}{2}, \qquad \mathfrak{S} = \frac{T'' + T}{2}$$

and half differences

$$\mathfrak{u} = \frac{\delta'' - \delta}{2}, \qquad \mathfrak{U} = \frac{T'' - T}{2},$$

and introducing the auxiliary magnitudes $\mathfrak{r}$ and $\mathfrak{w}$ determined by the conditions

$$\mathfrak{r}\cos\mathfrak{w} = \sin\mathfrak{U}\cot\mathfrak{u}, \qquad \mathfrak{r}\sin\mathfrak{w} = \cos\mathfrak{U}\tan\mathfrak{s},$$

we find the equation

$$\text{(II)} \qquad -\tan\varphi = \mathfrak{r}\sin(\mathfrak{S} + \mathfrak{w}).$$

By division of II and I we obtain the *sine ratio of the two unknown angles* $(\mathfrak{S} + \mathfrak{w})$ and $[S + w]$,

$$\text{(III)} \qquad \frac{\sin(\mathfrak{S} + \mathfrak{w})}{\sin[S + w]} = \frac{r}{\mathfrak{r}}.$$

However, since the difference

$$(\mathfrak{S} + \mathfrak{w}) - [S + w] = \frac{T'' - T'}{2} + \mathfrak{w} - w$$

of these angles is known, it is easy to calculate the *sum* of the angles by applying the sine tangent theorem (No. 40) to (III). From the sum and the difference we obtain directly the angles $\mathfrak{S} + \mathfrak{w}$ and $S + w$ themselves and consequently also the unknown angles

$$\mathfrak{S} = \frac{T'' + T}{2} \quad \text{and} \quad S = \frac{T' + T}{2}.$$

From S and the known difference $T' - T$ we then obtain the *sought-for time angles* T and T'; from $\mathfrak{S}$ and the known difference $T'' - T$ we obtain in similar fashion the time angles T and T''. By adding the right ascension to the time angle we finally obtain the *moments of the observations in sidereal time.*

The *sought-for latitude* then follows from (I) or (II), the *sought-for altitude* h from (1), (2), or (3).

NOTE. If the latitude is to be determined from *two* observations of the same star altitude and the time interval between them, we have at our disposal only equations (1) and (2) and must assume that the time angle T for one of the observations is known. Equation (I), all the magnitudes on the right side of which are known, then gives φ.

A remarkable special case of this situation is the

PROBLEM OF RICCIOLI: *From the time between the culminations of two known stars that rise or set at the same time, find the latitude of the observation point.*

This problem posed by Riccioli in 1651 is especially noteworthy in that the method employed makes possible determinations of latitude without an angle-measuring instrument.

If T and T' are the time angles of star risings, their difference $2U = T' - T$ is also the time between their culminations. Our initial equations (1) and (2) are simplified here (because $h = 0$) to

$$\cos T = \tan \delta \tan \varphi \quad \text{and} \quad \cos T' = \tan \delta' \tan \varphi.$$