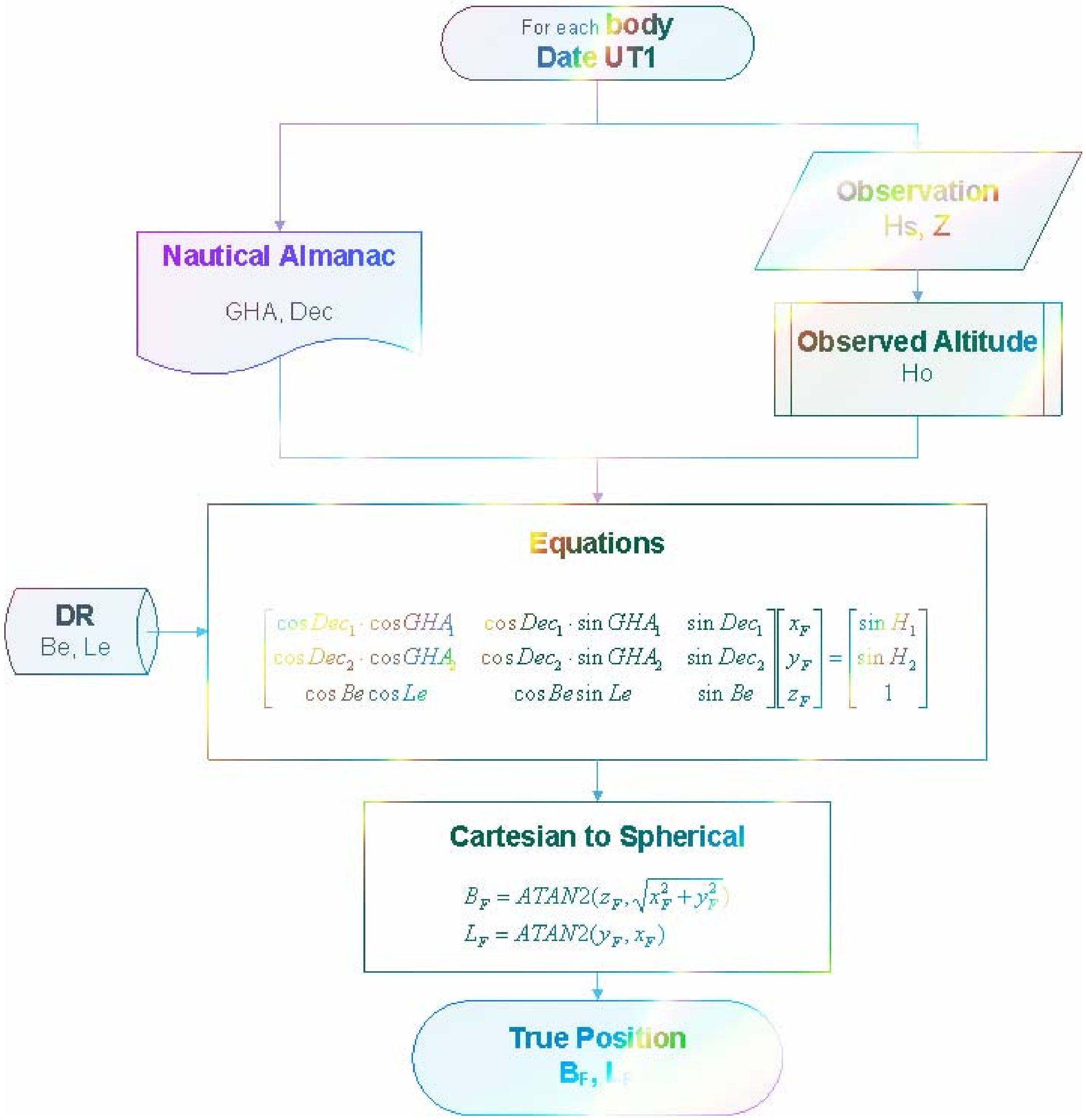
NAVIGATIONAL ALGORITHMS
Sight Reduction with Matrices



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Índice

Abstract

An analytical method for obtain the position by celestial sights using matrix algebra is described. The simplicity of the process allows the use of a hand held calculator or PDA.

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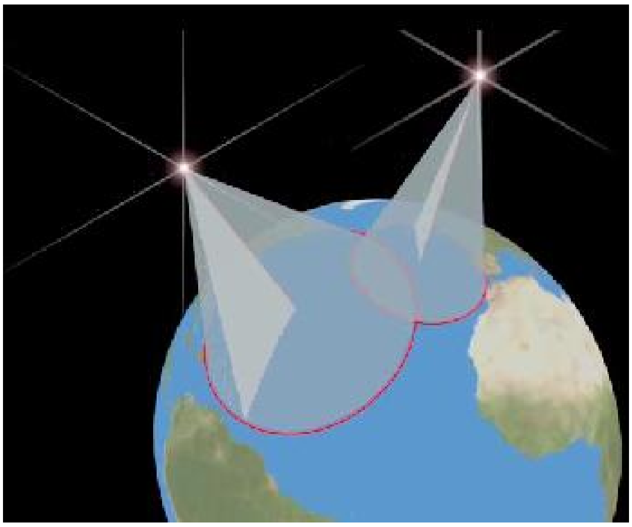
Finding position by stars

If we take some sight with a sextant, we shall be able to obtain our position by intersecting the associated circles of position, COP.



Circles of Equal Altitude

For one sight, the vectorial equation of the circle of equal altitude is:

$$\vec{O}P \bullet \vec{G}P = \cos(90^{o} - Ho)$$

And in Cartesian coordinates we have:

$$\{OP\} = \begin{bmatrix} \cos B \cdot \cos L \\ \cos B \cdot \sin L \\ \sin B \end{bmatrix}$$

$$\{GP\} = \begin{bmatrix} \cos Dec \cdot \cos GHA \\ \cos Dec \cdot \sin GHA \\ \sin Dec \end{bmatrix}$$

Where OP is the observer's position, GP the geographical position of the celestial body, GHA the Greenwich Hour Angle, Dec the declination, and Ho the celestial body's observed altitude.

The position is obtained by solving the system of linear equations of each COP for the Cartesian coordinates, and transforming them to spherical ones.

Take care with the signs of the variables; the mathematical functions use a different criterion to the nautical one:

- -90 ≤ B ≤ 90º [+N/-S]
- 0 ≤ L ≤ 360º [+ W ⇨ E]
- -90 ≤ Dec ≤ 90º [+N/-S]
- 0 ≤ GHA ≤ 360º [+ W ⇨ E]

Here is supposed that all observations are obtained simultaneously, if not the motion of the vessel and the motion of the bodies may be incorporated into the algorithm.

Two Observations

If two bodies are shot there are two solutions or one.

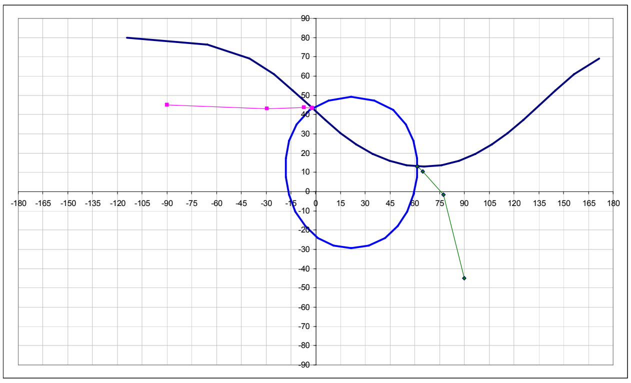
- Two solutions: The two COPs have two points of intersection. Our DR position determines what the real one is.
- One solution: if the two COPs are tangents. This theorist case is very improbable in navigation

There are three unknown parameters, the Cartesian coordinates of the true position, hence three equations are needed in order to calculate them: the two COP equations and the fact that from the observer's position the zenith altitude is 90º.

The algorithm is described in the appendix.

Iteration for the assumed position

If the assumed position is far away from the true one, solving the matrix system it obtained a bad solution for the position. Then an iterative process can improve the solution setting new assumed position equal to solution and solving once again until the error is acceptable.



2 COP and solution points in an iterative process for the two intersections

3 Observations

For three sights the system of equations is determined, and is it possible to obtain an explicit solution:

	Body	Ho	GHA	Dec
1	Body1	Ho1	GHA1	Dec1
2	Body2	Ho2	GHA2	Dec2
3	Body3	Ho3	GHA3	Dec3

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \frac{1}{\sin Ho_k} \begin{bmatrix} \cos Dec_k \cdot \cos GHA_k \\ \cos Dec_k \cdot \sin GHA_k \\ \sin Dec_k \end{bmatrix}$$

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \begin{bmatrix} x_F \\ y_F \\ z_F \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_F = \frac{|Mx|}{|M_3|} \qquad y_F = \frac{|My|}{|M_3|} \qquad z_F = \frac{|Mz|}{|M_3|}$$

$$|M_3| = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$|M_x| = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}$$

$$|M_y| = \begin{vmatrix} x_1 & 1 & z_1 \\ x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \end{vmatrix}$$

$$|M_z| = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

```
M3 = x1*y2*z3+y1*z2*x3+z1*x2*y3-
(z1*y2*x3+x1*z2*y3+y1*x2*z3)

Mx = y2*z3+y1*z2+y3*z1-
(y2*z1+y1*z3+y3*z2)

My = z2*x3+z1*x2+z3*x1-
(z2*x1+z1*x3+z3*x2)

Mz = x2*y3+x1*y2+x3*y1-
(x2*y1+x1*y3+x3*y2)
```

This method works only for one point of intersection. Otherwise a least squares method must be used to find the most probable celestial position.

n Observations

The matrix system is overdetermined, and a least squares solution for the fix may be used.

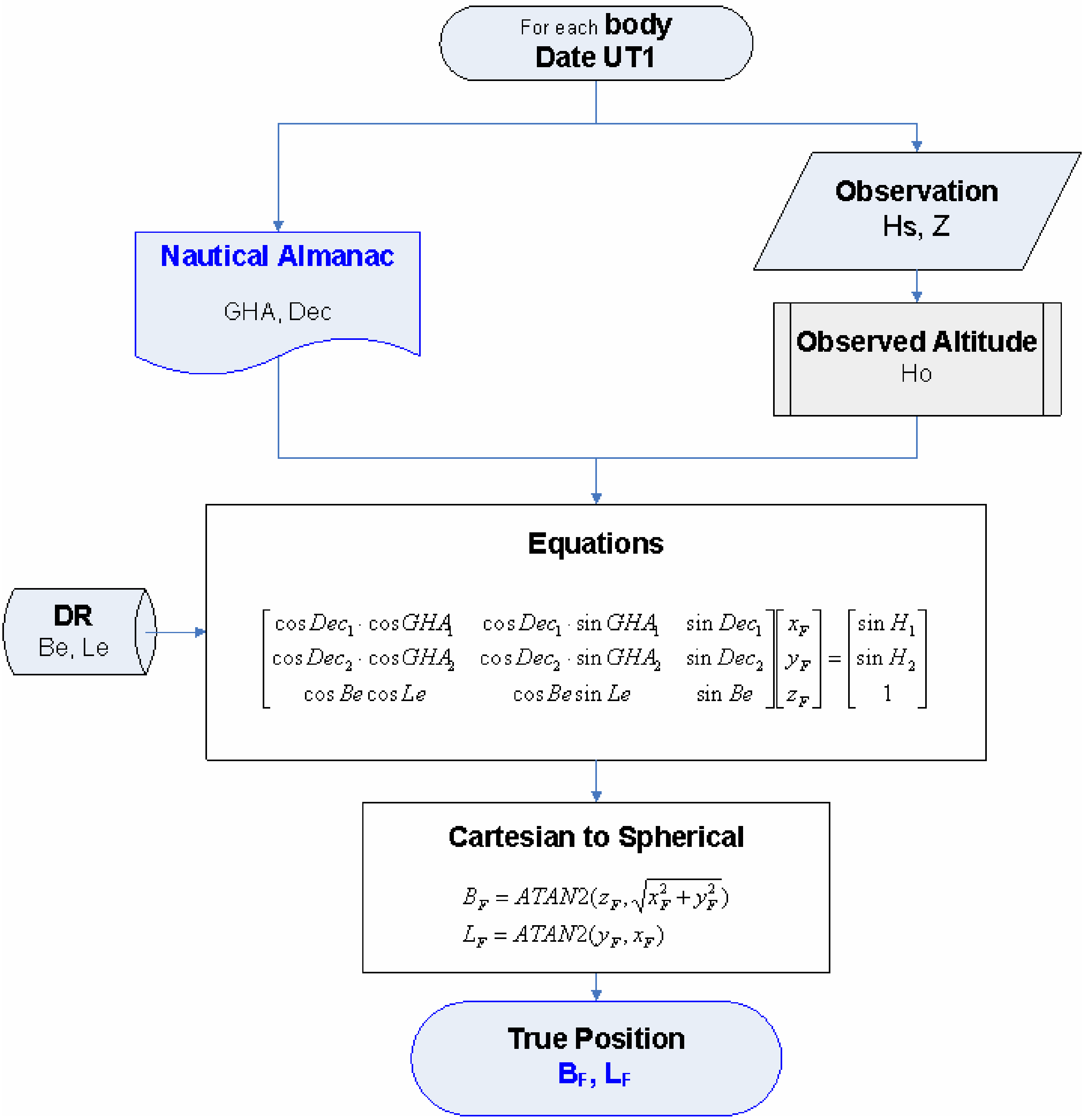
For n ≥ 3 observations it is possible to incorporate the assumed position by doing Dec = B, GHA = L, Ho = 90º, but may be ignored.

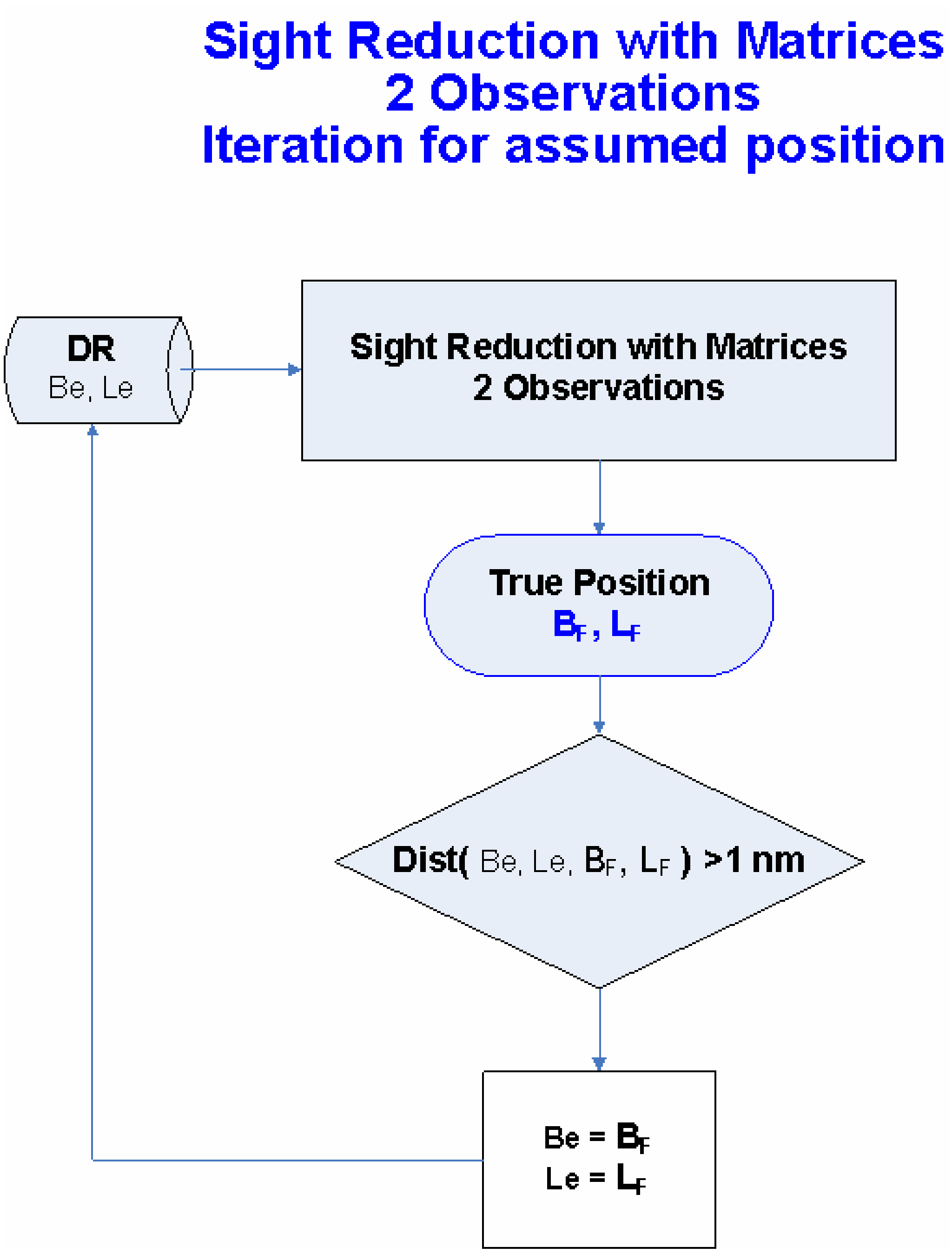
The details of the algorithm are described in the appendix.

A1. Algorithms

Sight Reduction with Matrices

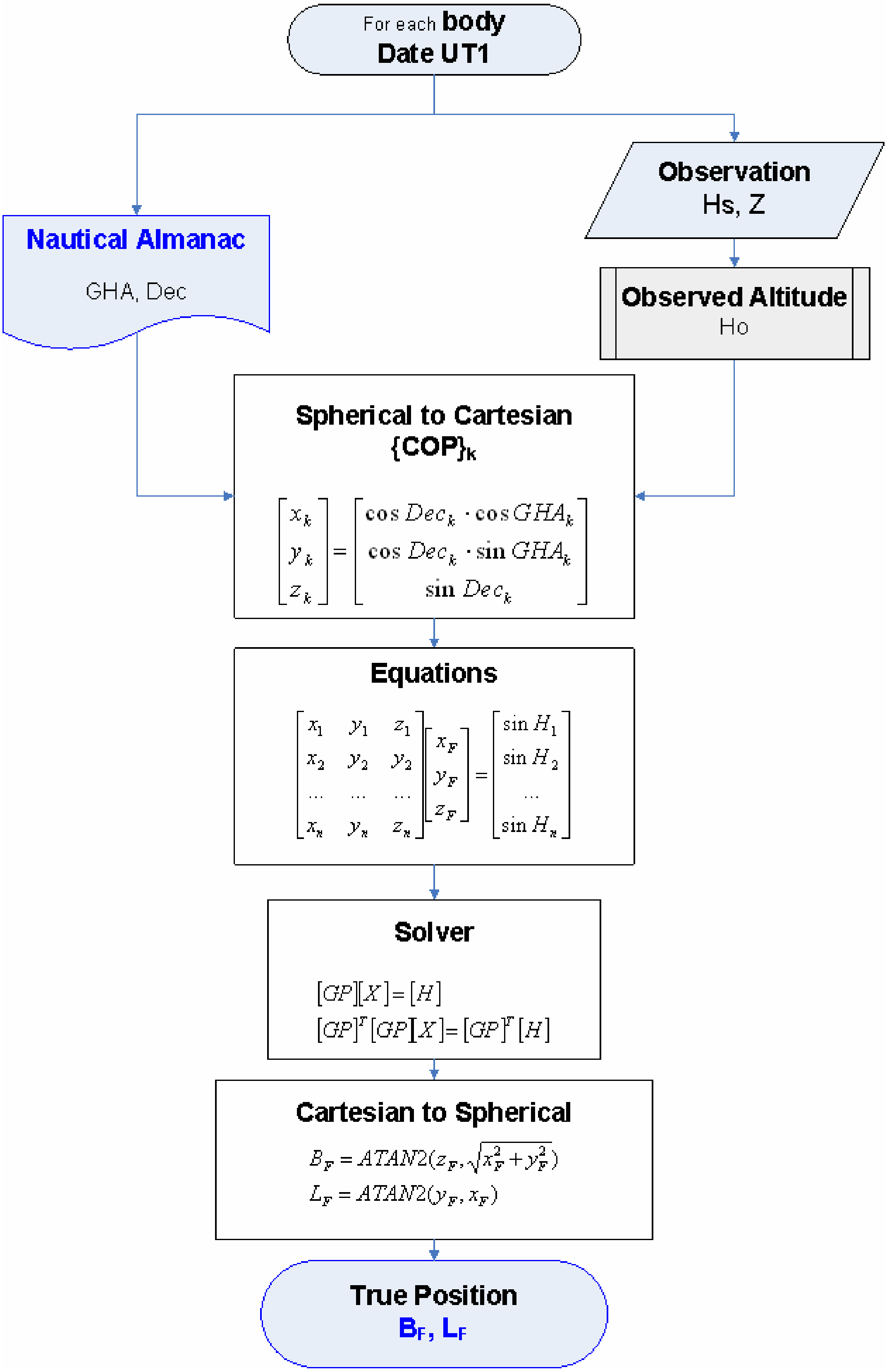
2 Observations





Sight Reduction with Matrices
2 Observations
Iteration for assumed position
DR
Be, Le
Sight Reduction with Matrices
2 Observations
True Position
B_F , L_F
Dist(Be, Le, B_F, L_F) >1 nm
Be = B_F
Le = L_F

Sight Reduction with Matrices
n Observations



A2. Examples

3 Observations

Using the method explained in the *3 Observations* section, not the general one.

Input data:

K	Body	Ho	GHA	Dec
1	Arcturus	22.2	166.829	19.335
2	Vega	70.15	115.277	38.758
3	Dubhe	20.35	226.98	61.909

The equations in matrix form are:

```
| -2.432  0.569  0.876 | |x|   | 1 |
| -0.354  0.750  0.666 | |y| = | 1 |
| -0.924 -0.990  2.537 | |z|   | 1 |
```

Solving the system:

```
|x| | 0.006|
|y|=| 0.731|
|z| | 0.682|
```

And the position y latitude and longitude coordinate is:
Fix(42.98208, 89.56648)

Sight Reduction with matrices

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Assumed Position

Be = **43.3167** °
Le = **2.0000** °

Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	**Enif**	**9.897**	**338.392**	**50.775**
2	**Schedar**	**56.560**	**294.303**	**46.438**

Equations in matrix form

$$\begin{vmatrix} 0.91588803 & -0.36277983 & 0.17187179 \\ 0.22679974 & -0.50222783 & 0.83446335 \\ 0.72713001 & 0.02539194 & 0.68603002 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.77466864 \\ 0.72463308 \\ 1 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.72712784 \\ 0.02538372 \\ 0.68603263 \end{vmatrix}$$

Solution in Geographic coordinates

B = **43.3168721** ° **46.6831279**
L = **1.99935875** ° **88.0006412**

Iteration

Set assumed position = solution and solve once again

Sight Reduction with matrices
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Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	**arcturus**	**19.335**	**166.829**	**22.200**
2	**vega**	**38.758**	**115.277**	**70.150**
3	**dubhe**	**61.9090**	**226.9800**	**20.350**

Equations in matrix form

$$\begin{vmatrix} -0.9187769 & 0.21500662 & 0.33109087 \\ -0.33296938 & 0.70513465 & 0.62603236 \\ -0.32125502 & -0.34426282 & 0.88220084 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.37784079 \\ 0.94058481 \\ 0.34775398 \end{vmatrix}$$

$$\begin{vmatrix} 1.05822439 & -0.3217352 & -0.79605969 \\ -0.3217352 & 0.66195961 & 0.20891489 \\ -0.79605969 & 0.20891489 & 1.279816 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} -0.77205504 \\ 0.62475844 \\ 1.02072501 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.00553388 \\ 0.73137457 \\ 0.68160992 \end{vmatrix}$$

Solution in Geographic coordinates

B = **42.9820851** °
L = **89.5664846** °

Sight Reduction with matrices

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Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	**Mars**	**22.393**	**72.777**	**29.518**
2	**Alioth**	**55.928**	**357.303**	**77.542**
3	**Alkaid**	**49.2830**	**343.9330**	**76.027**
4	**Arcturus**	**19.1480**	**336.8970**	**57.045**

Equations in matrix form

$$\begin{vmatrix} 0.273764 & 0.88313346 & 0.38095742 \\ 0.55961371 & -0.02636135 & 0.82833422 \\ 0.62684273 & -0.18053781 & 0.75794082 \\ 0.86891246 & -0.37067635 & 0.32800942 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.49269697 \\ 0.9764544 \\ 0.97040962 \\ 0.83909807 \end{vmatrix}$$

$$\begin{vmatrix} 1.53605491 & -0.20823613 & 1.32796078 \\ -0.20823613 & 0.95061449 & 0.05617793 \\ 1.32796078 & 0.05617793 & 1.5133306 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 2.01871695 \\ -0.07685291 \\ 2.0072723 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.71873376 \\ 0.03556087 \\ 0.69437855 \end{vmatrix}$$

Solution in Geographic coordinates

B = **43.977596** °

L = **2.83251973** °

Sight Reduction with matrices

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Geographic position & Observed Altitude

n	Body	Dec	GHA	Ho
1	**Mars**	**22.393**	**72.777**	**29.518**
2	**Alioth**	**55.928**	**357.303**	**77.542**
3	**Alkaid**	**49.2830**	**343.9330**	**76.027**
4	**Arcturus**	**19.1480**	**336.8970**	**57.045**
5	**Polaris**	**89.2900**	**151.6270**	**43.370**

Equations in matrix form

$$\begin{vmatrix} 0.273764 & 0.88313346 & 0.38095742 \\ 0.55961371 & -0.02636135 & 0.82833422 \\ 0.62684273 & -0.18053781 & 0.75794082 \\ 0.86891246 & -0.37067635 & 0.32800942 \\ -0.01090296 & 0.00588857 & 0.99992322 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.49269697 \\ 0.9764544 \\ 0.97040962 \\ 0.83909807 \\ 0.68670698 \end{vmatrix}$$

$$\begin{vmatrix} 1.53617379 & -0.20830033 & 1.31705866 \\ -0.20830033 & 0.95064917 & 0.06206605 \\ 1.31705866 & 0.06206605 & 2.51317705 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 2.01122981 \\ -0.07280919 \\ 2.69392655 \end{vmatrix}$$

Solution in Cartesian coordinates

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix} = \begin{vmatrix} 0.7187278 \\ 0.03555919 \\ 0.69438517 \end{vmatrix}$$

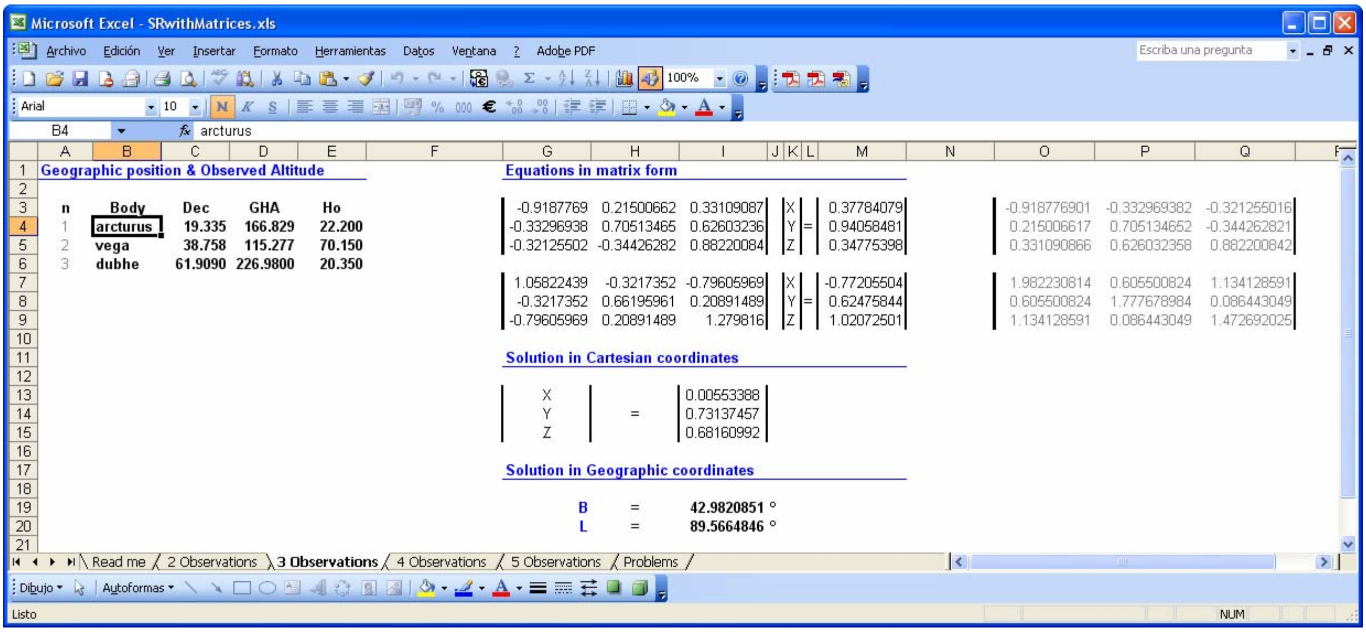
Solution in Geographic coordinates

B = **43.9781085** °

L = **2.83240931** °

A3. Software

An Excel sheet is available for up to 5 observations



A4. Source code

The algorithms are implemented in the Excel sheet

A5. References

- Watkins. R. and Janiczek. P. M., Sight Reduction with Matrices, NAVIGATION, Journal of The Institute of Navigation, Vol. 25, No. 4, Winter 1978-79, pp. 447-48
- Navigational Algorithms - Vectorial equation of the circle of equal altitude. Andrés Ruiz, http://www.geocities.com/andresruizgonzalez