The Attainment of Precision in Celestial Navigation

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There are two approaches to the study of errors in astronomical observations at sea. One is to collect as much information as one can from practising navigators; the other, described here, is to carry out controlled experiments with the object of breaking down the various *causes* of error, a virtually impossible task if the first approach is adopted. In both cases a large mass of data must be carefully studied and analysed, and this is particularly important where one aims to resolve the errors into separate components (those due to dip uncertainties, horizon, body, degree of twilight, and sea state).

A total of 500 observations is not really adequate to discharge this task to complete satisfaction and it would be rash to place too much reliance on tabulations of systematic and random errors based on, at worst, a single string of as few as nine observations. The results presented here are, however, of considerable interest, even though a more valuable survey would have been made if a far larger number of observations, preferably taken by many different observers and in different environmental conditions, had been available.

1. INTRODUCTION. The reduction of observations of celestial bodies made at sea to a line of position is subject to error from numerous and, sometimes, subtle causes. Properties of the atmosphere, state of the sea, optical and mechanical characteristics of the sextant, and psychological-physiological factors all enter. So long as one is concerned only with rough navigation—getting from one port to another on a reasonable track—approximate corrections for the various sources of error in celestial sights suffice. The marine sextant is, however, a remarkably sensitive and precise instrument and where precise navigation is required, as in oceanographical research, it is desirable to attain the full precision of which the instrument is capable. One is rarely afforded the opportunity of studying navigational errors at sea while field conditions cannot be simulated with much success in the laboratory. This paper describes studies of celestial navigation made in coastal waters where a deliberate attempt is made to separate and study individually the various sources of error which enter precise celestial navigation.

A number of investigators have considered the magnitude of the errors which may be expected in the practice of celestial navigation at sea[1–5], but do not separate out the various causes of error. In addition to this work, Shufeldt[6] has reported on a series of experiments dealing with the improvement of instruments for celestial navigation, particularly the use of high magnification telescopes on the marine sextant.

2. METHOD OF THE EXPERIMENTS. In modern practice, celestial navigation consists of observing the altitude h of a celestial body above the horizon at a given instant of time and computing from this observation a line of position. Other methods, such as determining a fix from simultaneous observations of h and dh/dt, the noon sight, or the time sight, have either proved unpractical or are special cases of the line of position method. A modern marine sextant of good quality will read to $0'.1$; if the full accuracy of the sextant could be utilized, lines of position located to within about 200 yards would be attained, an accuracy quite satisfactory even for research work at sea. By *navigational errors* is meant all factors which increase the error in a line of position above that just quoted.

A typical celestial navigation experiment will consist of a series of altitude observations made from a known position. The errors in the observations are determined by comparison with computed altitudes and are the same as the error which would appear in locating a line of position from the observed altitudes. These errors fall naturally into two groups, random and systematic. Random error is measured by the range of the individual observations about the mean and is given quantitatively by the standard deviation. The systematic error is the departure of the mean from the computed altitude and has been found, in the present work, to be in most cases constant over the period of ten minutes or so required to make a series of observations. The plan of the experiments, then, has been as follows: A series of observations, usually nine or more, was made under given conditions. The standard deviation of the observations, σ, and the mean observed altitude and its departure from the calculated altitude, Δh, were obtained. These were to be related to the conditions of observation: quality or distinctness of the horizon, atmospheric refraction (which determines the refraction correction to the altitude and influences the dip of the horizon), the extent of obscuration of the body observed by cloud cover, motion of the observer, and the characteristics of the instrument used, particularly the optical properties of the sextant telescope. Since the experiments could not be done in the laboratory, control over the conditions of observation was not possible. Instead, it was necessary to make observations on many different days in which one or another of the conditions of observation departed significantly from normal.

Observations were made from the shore over the waters of Long Island Sound near Guildford, Connecticut, and at sea on Long Island, Block Island, and Rhode Island Sounds. The shore observations were made at the following position: 41°16′06″N., 72°40′04″W. and were confined to azimuths between 120° and 190°.

For most of the observations a sextant of six-inch radius reading to ten seconds of arc manufactured by Henry Hughes & Son, Ltd., was used. It was carefully adjusted according to standard procedures. There is no provision for adjustment of the telescope axis, so to test for collimation error Δh was plotted against h for about 100 observations with h ranging from 10 to 86°. The points fell at random showing that, since collimation error is proportional to $\tan \frac{1}{2} h$, this error was not present. Telescopes having nominal magnifications of 2½, 3 and 8 were used. Particulars of the telescopes, as determined by measurement, are given in the following table.

Nominal magnification	Type	Measured magnification	Field of view	Diameter of object glass (mm.)
			° ′	
2½	Galilean	2·8	7 30	30
3	Galilean	3·3	5 44	43
8	Prism Monocular	8·0	3 45	30

A few observations were made with a standard U.S. Navy Mark II sextant.

The measurements of the dip of the horizon conducted from the shore were made with a specially constructed level. This consisted of a telescope with filar eyepiece set in V-blocks. The V-blocks were levelled with a precision bubble level sensitive to five seconds of arc. For timing observations a second-setting stop watch was used. The stop watch was set by an 'Accutron' electric watch which, in turn, was calibrated against the time signals broadcast by radio station CHU.

To determine Δh, the average of the observed altitudes must be compared to a computed altitude. It was found that the most satisfactory method of computing h is to use the relation

$$\sin h = \sin \phi \sin d + \cos \phi \cos d \cos t$$

where ϕ is the latitude, d the declination, and t the hour angle. Solutions to $0'.1$ can be obtained with about equal facility using tables of natural functions and a desk calculator or by logarithmic functions. An alternative method is to use the *Tables of Computed Altitude and Azimuths* (H.O.214, H.D.486) and the method of triple interpolation. This method is rapid but errors greater than $0'.5$ may occur when Δd or Δt are changing rapidly. It is not satisfactory to plot a line of position from a point which eliminates the ϕ and t corrections unless a stereographic rather than a Mercator projection is used for the plotting sheet. A corollary of this is that, in careful navigation, it is better to compute the altitude from the dead reckoning position, by the above equation, than from some assumed position which may simplify the calculations, or by the triple interpolation method of H.O.214.

Celestial coordinates are taken from the *Nautical Almanac*. The error arising from the limited accuracy of this tabulation has been analysed by Sadler[7] as follows: For stars the contribution to the standard deviation arising from errors in the almanac is $\pm 0'.04$, and for the Sun, is $\pm 0'.05$. These errors are negligible in the present work.

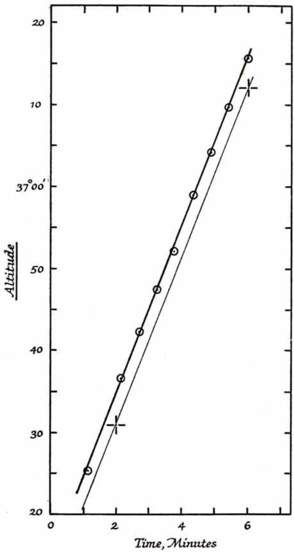

FIG. 1. A series of altitudes of the Sun taken with the Hughes sextant under good conditions. Calculated altitudes are shown by the crosses. Note that the difference between the observed and calculated altitudes is constant.

The method of data analysis employed is illustrated using the set of observations shown in Fig. 1. The observed altitudes are plotted as a function of time. The computed altitude, including corrections for index error and for dip of the horizon, refraction, and (when required) for semi-diameter from the *Nautical Almanac*, is also plotted. The deviation of each observed altitude from the computed altitude is obtained and from these deviations the average Δh is computed. The standard deviation, σ, of the points about Δh is then calculated. The standard deviation associated with Δh is then $\sigma/\sqrt{(n-1)}$ when n observations have been made, since the computed altitudes are taken to be exact. There occur from time to time in the observations one or two points in a given set which are considerably out of line with the other points. In deciding whether or not to retain these points in the analysis, Chauvenet's criterion is used: if the probability of the occurrence of such a deviant point was less than $1/(2n)$ it was rejected and a new Δh and σ are computed.

3. RESULTS OF THE EXPERIMENTS. In all, some 500 observations made under a variety of conditions have been analysed in the present work.

Index Error. To measure the altitude of a celestial body, two readings must be made on the arc of the sextant, one when the body is brought down to the horizon, the other when the index and horizon glasses are parallel, i.e., the index reading. Any error in fixing the index setting of the instrument appears directly in the measured altitude; accurate determination of the index error is, therefore, of primary importance. Unfortunately, it is often difficult to do this with the requisite precision. It is not sufficient to make a single, accurate index determination and then assume that this value will hold for future observations. With the Hughes sextant, for example, no index adjustments were made over a period of five months and the index error was found to vary over a range of 2′.

Several methods of determining the index error are available:

(i) To superimpose the direct and reflected images of a star. Satisfactory results are not often obtained by this method. Because of the finite resolving power of the telescope and defects in the optical system of the sextant, the star image is not perfectly sharp. With the 2½× telescope on the Hughes sextant, a star image will superimpose over a range of 5′. To obtain a reliable index error it is necessary to find the mean of readings made as one image is brought alternately up and down. Experience shows, however, that it is difficult to judge the point of tangency of imperfect images and the error in the readings is unacceptably large. This same difficulty limits the utility of measuring inter-stellar angles as a means of checking the accuracy of a sextant.

(ii) To align the direct and reflected images of the horizon. The width of the diffraction maximum from a line source is less than that from a point source in the ratio of 1 : 1·22. Hence it is anticipated that a more accurate index setting can be made by the horizon method. This is verified by experiment but the improvement is much greater than suggested by the above ratio, which is calculated from diffraction theory. Evidently the eye finds it much easier to judge the continuity of a line than the superposition of points.

The standard deviation of some 165 determinations of index error by the horizon method was calculated with the results shown in Table I.

TABLE I. STANDARD DEVIATION OF INDEX ERROR BY HORIZON METHOD

Horizon Quality	σ
	′
Excellent or very good	0·28
Good or Fair	0·35
Poor	0·50

The accuracy of the index error determination depends to some extent on the quality of the horizon available. Horizon quality is rated on an arbitrary scale from excellent to poor. An 'excellent' horizon appears absolutely sharp through any of the telescopes used and occurs only when the atmosphere is unusually clear and there are sufficient small waves on the water surface to give sharp contrast between water and sky. A 'poor' horizon occurs when the boundary between sky and water is hardly distinguishable; it may be due to low contrast, as on an overcast day with high visibility but wind (and consequently, waves) lacking, or to poor visibility in haze or fog. No indication of variation in the magnitude of index error with horizon quality was detected.

The importance of the size of the standard deviations listed in Table I for a sextant altitude reading is that the probability of a single index determination being in error by more than σ is 32 per cent. The probability of the error being greater than $2{\cdot}6\sigma$ is one in 106. Repeated readings must therefore be made in order to obtain a reliable index determination. Twenty-five readings on a poor horizon are required to fix the index error to within $\pm 0\overset{\prime}{.}26$ if one in a hundred is the greatest probability of a deviation larger than this which can be tolerated.

Experience shows that, with the horizon method, best results are obtained using the sextant sight tube rather than a telescope, and that a high power telescope is almost useless in this application. As the reflected image of the horizon is brought down to the direct image, the two images will appear to fuse suddenly into one line; a similar phenomenon occurs as the reflected image is brought up. The index error is the mean of the two readings taken when the images just fuse. High magnification does not help the observer judge this point or reduce the angular interval in which the two images remain fused.

(iii) To make the direct and reflected images of the Sun tangential. In this method the upper limb in the reflected image is made tangential to the lower limb in the direct image and then the images are reversed. The index error is the mean of the readings. A telescope of high magnification fitted with a dark eye cap is used. The eye is capable of judging the tangency of two discs with great sensitivity because it can average along the relatively long, sharply defined perimeter of the Sun. A careful observer making observations ashore on a clear day can repeat tangency settings to the least count of the instrument and so fix the index error with great accuracy. Unfortunately, it is rather difficult to use this method on a moving vessel unless the Sun is at low altitude, when the method is not reliable due to distortion of the Sun's image.

The tangency method provides a very convenient means of checking the shades of the sextant for prismatic error; the presence of such error is indicated by a difference in reading obtained when the eye cap is removed and appropriate shades are inserted.

In examining the index determinations for the Hughes sextant made over a period of five months, no evidence of any systematic difference in the index error determined by the horizon and tangency methods was found. There is also no correlation between index error and either air temperature or the presence or absence of sunlight, indicating freedom from thermal distortion in the frame of the instrument.

In summary, it is seen that uncertainty in the index error of the magnitude given in Table I will be one important component in the uncertainty of any altitude measurement with the marine sextant. This has not always been recognized in previous studies of navigational errors. It may be that some or all of what is called 'personal error' in earlier studies[4] is, in fact, variable index error.

4. Random errors. Table II shows the random errors for observations of the Sun and Moon during daylight, all made using the standard sextant telescopes (2½× on the Hughes sextant, 3× on the Navy sextant). The random error is expressed as the standard deviation, σ, of an individual observation. Also given is the ratio R/n of the number of points rejected by Chauvenet's criterion to the total number of points taken, a measure of the frequency of occurence of 'wild' observations.

Table II. Random error of daylight observations, standard telescope

Conditions of observation	σ	R/n
SUN	′	
Ideal conditions	0·33	2/39
Cloud cover only	0·32	1/15
Motion only (small boat)		
Yawing at anchor	0·32	0/9
Reaching, Force 3 wind	0·31	1/9
Beating, Force 4 wind	1·6	0/6
Horizon only		
Good	0·52	0/9
Poor	0·70	0/38
Horizon + motion		
Poor horizon, Force 2 wind, small boat	0·43	3/18
Good horizon, Force 3 wind, large boat	0·44	1/42
MOON		
Ideal conditions	0·40	0/19
Horizon only		
Fair–Good	0·42	0/28

Ideal conditions in Table II refers to observations made on shore with an excellent horizon and no cloud cover. The remaining observations are classified on the basis of one or more conditions departing from the ideal. The data for 'cloud cover only' were taken as stratus clouds advanced before rain; about half were made when the limb of the Sun could not be truly distinguished. It is seen that this obscuration of the Sun results in no increase in σ.

The observations where 'motion only' was a factor were all taken aboard a 16-ft. sloop sailed on the open waters of Long Island Sound with south-westerly winds blowing over a fetch of some 30 miles. The boat was sailed along a given compass course and frequent bearings on shore points were taken to fix her track. Positions to the nearest 0'.1 from the plotted track were used in working out the calculated altitudes. The principal obstacles to obtaining good data under these conditions were found to be, first, the relatively rapid yawing of the small boat (even at the hand of a skilled helmsman) and, second, the amount of spray coming aboard. The first makes it difficult to swing the sextant arc properly as the Sun cannot be kept sufficiently long in the field of view, the second results in image deterioration. The results show that fully accurate observations can be made from a small boat until spray makes it impracticable to use the sextant. Observations made aboard a 53-ft. schooner (designated 'large boat' in Table II) were found to be of accuracy comparable to that attained in the small boat but were much easier to make.

With tolerable conditions for observation, horizon quality is the most important single factor influencing the standard deviation of daylight observations, but even here the increase in σ under poor conditions is hardly significant. When there is both motion of the observer and a poor horizon, there is a tendency for more 'wild' observations to occur. In practical navigation, however, the effect of increased σ and R/n can be largely overcome by using the average of a sufficiently large number of observations rather than a single observation.

Observations of the Moon in daylight show no greater random error than those of the Sun.

In summary of the daylight observations, it seems safe to conclude that the standard deviation of a sextant altitude made in daylight is 0'.4 and increases only slightly under poor observing conditions. This accuracy is comparable to that reported by Smiley[1] who gives the probable error of a single observation made with a 3× telescope as 0'.25 ($\sigma = 0'.37$). In an earlier paper[8] he reports probable errors ranging from 1'.6 to 0'.09 for series of Sun sights made in a study of refraction at low altitudes. There is always a certain probability of wild observations occurring and this increases under difficult conditions. Thus, in careful navigation, the average of several (preferably five or more) observations should always be used in preference to a single observation, regardless of how carefully the latter is made, for experience shows that wild observations cannot be discovered reliably at the time of observation.

Special Telescopes. A number of daylight observations were made with the 3× and 8× telescopes on the Hughes sextant. The characteristics of these instruments significant for sextant use are their superior light gathering power as compared with the standard telescopes and their resolving power, only the latter being important for daylight observations. It is the resolving power of the telescope as mounted on the sextant which is important rather than that obtained in a laboratory test of the telescope alone. The resolving power of the complete optical system was determined for each telescope by measuring the diameter of the image of the star Altair. This is done by bringing the direct and reflected images tangent first with the reflected image below and then above the direct image, and noting the difference in readings on the arc. Results of this test for the telescopes used on the Hughes sextant are given in Table III. The theoretical resolving power, θ_T, is calculated in each case from the relation

$$\theta_T = 14{\cdot}1/a$$

where θ_T is in seconds of arc when a, the aperture limiting the beam forming the primary image, is in centimetres; θ_P is the practical resolving power as determined from measurement. The full, theoretical resolving power of the 2½× and 3× telescopes could not be realized in any case because this magnification is too low (less than the normal magnification). In fact, the much better θ_P of the 8× monocular must be due largely to its higher magnification. The entry 3×, OF refers to the 3× telescope deliberately set out of focus as used in some of the tests to be described.

TABLE III. RESOLVING POWER OF SEXTANT WITH VARIOUS TELESCOPES

Telescope	θ_T	θ_P
	″	′ ″
2½×	4·5	2 20
3×	3·4	1 30
3×, *OF*	—	12
8×	4·7	0 30

The random errors obtained during daylight observations with the special telescopes are recorded in Table IV. The first conclusion to be drawn from the data is that the use of these special telescopes results in no very marked improvement in σ. The poorer results obtained under ideal conditions with the 3× telescope are ascribed to the inferior optical quality of this instrument, particularly to its tendency to show colour fringes on a sharp horizon. With a 'poor' horizon the colour contrast between water and sky is not so great and the detrimental effect of this chromatic observation is not as important. With the poor images obtained when the 3× telescope is set out of focus, σ is considerably increased. This demonstrates the need for high quality construction in sextant telescopes. It had been anticipated that under conditions of small boat motion it would be difficult to use the 8× monocular and that a high σ would be obtained. This was expected on the basis of the smaller field of view afforded by this instrument: the principal difficulty in making sextant observations from a small boat is keeping the Sun and horizon in view for long enough to swing the arc properly. It was found, however, that the σ for the 8× data taken from a small boat is comparable to that obtained with the standard telescope.

Shufeldt[6] has reported very substantial improvements in the reproducibility of sextant observations taken with high-power telescopes. He reports deviations of individual observations from the mean in terms of an average deviation rather than σ, but σ can be calculated from the average deviations to a fair approximation by multiplying by 1·25. For 20× magnification Shufeldt reports σ ranging from 0′·05 to 0′·26 for various conditions of observation whereas for 6× magnification his extreme range of σ is from 0′·09 to 0′·89. No average deviations are reported for the 3× Navy telescope but it is evident from his Fig. 1 that very poor reproducibility was obtained with that instrument. In contrast, the data reported here show little or no advantage for the high-power telescope but better results than Shufeldt's with the standard, low-power telescope.

TABLE IV. RANDOM ERROR OF DAYLIGHT OBSERVATIONS. SPECIAL TELESCOPES

Telescope	Conditions of observation	σ	R/n
(×)	Ideal conditions	′	
8		0·37	0/27
3		0·52	0/9
	Motion only, small boat		
8	Reaching, Force 3 wind	0·40	0/9
8	Beating, Force 4 wind	3·6	0/9
3, *OF*	Reaching, Force 3 wind	1·5	0/9
	Horizon only		
8	Good	0·35	0/18
3	Good	0·7	0/9
8	Poor	0·92	0/9
3	Poor	0·22	1/9
	Horizon + motion, small boat		
8	Poor horizon, beating, Force 2 wind	0·8	2/9
3	Poor horizon, beating, Force 2 wind	0·35	0/18

First, with regard to the better performance of the 2½× telescope in contrast to the 3× standard Navy telescope, this difference is believed to be due to the superior optical quality of the 2½× instrument. The importance of optical quality was demonstrated above in the experiments with the 3× telescope. Many of the standard Navy telescopes are of inferior practical resolving power whereas most higher-power telescopes used in navigation experiments are of special construction and are inherently high quality instruments. It is suggested that some of the increased performance previously ascribed to high magnification is actually due to better optical quality as manifested by freedom from aberration and high resolving power.

Reproducibilities comparable to the best reported by Shufeldt were never realized in the experiments reported here. A number of factors are probably responsible for this: rather than specially constructed instruments an ordinary, production model, sextant was employed. Its condition was comparable to that which might be expected for a reasonably well cared for sextant in regular use on board ship. Thus the mirrors, although in very good condition, were not perfect. The smallest division of the Hughes sextant used was 10″ so that a σ of ±0′.06 was introduced at once in the reading of the instrument. Almost all the observations were taken by a single observer alone so that a timing error of $\pm\frac{1}{2}^{s}$ might be expected. This could lead to a contribution to σ of ±0′.1. Standard deviations smaller than 0′.15 could not, therefore, be expected in this work. This is not an important limitation, however, because it is substantially smaller than the standard deviation of the index error (which must always be included when the deviation of a line of position is being calculated) and, as will be shown later, is substantially less than the uncertainties due to systematic errors.

In conclusion, it appears that the use of high-power telescopes does not offer any substantial advantage in daylight observations but that it is desirable to have a high degree of optical quality (high resolution and freedom from aberration) in the entire optical system of the sextant.

Dawn and Dusk Observations. At dawn or dusk the illumination of the horizon changes continually while observations are being made and it is reasonable to suppose that the random and systematic error in the altitude will likewise change. To allow for this, a somewhat different method of calculating σ and Δh is

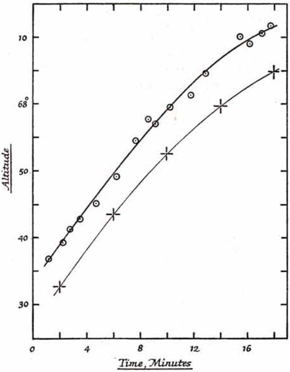

FIG. 2. A series of altitudes of Arcturus taken with the Hughes sextant during evening twilight. Calculated altitudes are shown by crosses. Δh is not constant for twilight observations.

used than for the daylight observations. A typical set of star altitudes obtained during evening twilight is shown in Fig. 2. A curve of best fit is drawn through the points by eye; Δh can be read off for any time. To obtain σ, the points are divided in groups, usually of nine each, and the σ of the group relative to the curve of best fit is calculated.

Some means of specifying the state of darkness on the horizon is needed. All of the observations in this set of experiments were made in an arc of azimuth extending about 20° to either side of 140° true bearing. To a good approximation, then, the darkness of the horizon is measured by the quantity t^*,

$$t^* = \frac{t - t_{ss}}{t_{nt} - t_{ss}}$$

where, for evening observations, t_{ss} is the time of sunset, t_{nt} is the time of the end of nautical twilight, and t is the time of observation. At sunset $t^* = 0$ and at the end of nautical twilight $t^* = 1$. The values of t_{ss} and t_{nt} used were obtained from data in the *Nautical Almanac* corrected for the longitude of the place of observation and reduced to local zone time. Data for large t^* were taken on nights when there was no Moon.

The other relevant factor is horizon quality; this is rated on the same scale as was used for the daylight observations. Cloud cover has not been found to be a significant factor except in so far as haze may delay the start of observations in the evening or a partly cloudy condition may interrupt the observations. The effect of motion of the observer on the standard deviation of star observations has not yet been specifically tested.

The standard deviations found in some 250 star observations range from $1'.6$ to $0'.24$ and average $0'.58$. This includes data taken with the $2\frac{1}{2}\times$, $3\times$ and $8\times$ telescopes on the Hughes sextant; there was no significant difference in the reproducibility obtained between the three. Many of the observations were taken in sets commencing as soon as the appropriate star or planet became visible and extending until the horizon could no longer be distinguished or, in the case of the $8\times$ monocular, until total darkness had set in. The stars observed ranged in brightness from Vega (magnitude 0·1) to Alphecca (magnitude 2·3) and included the planets Mars and Saturn. No correlation between σ and the magnitude of the observed body was found although, of course, the dimmer stars can only be observed at relatively large t^* values.

With the $2\frac{1}{2}\times$ telescope it is found that, on azimuths about 140°, $t^* = 0{\cdot}55$ is about the limit for good observations. As the horizon becomes darker, it simply is no longer seen with this telescope and σ increases rather suddenly to a large value. With the $3\times$ telescope, which has a much larger objective, observations can be continued to about $t^* = 0{\cdot}7$. With the $8\times$ monocular it was found possible to distinguish the horizon on clear nights with no Moon well after the end of nautical twilight. There is, however, a substantial increase in σ as shown in Fig. 3. The observations on which this figure is based were made on the star Altair;

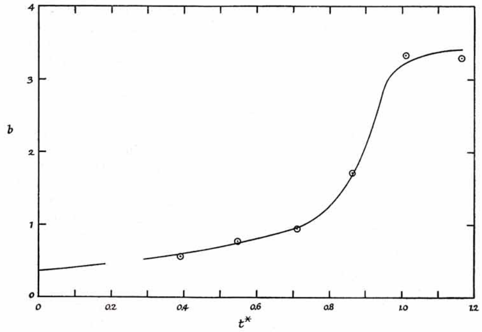

FIG. 3. The standard deviation of an individual observation of Altair during evening twilight. Altitudes were taken with the 8 × 30 monocular mounted on the Hughes sextant. At sunset $t^* = 0$; at the end of nautical twilight, $t^* = 1 \cdot 0$.

at about $t^* = 0 \cdot 7$ it was necessary to insert a shade glass in order to prevent the bright, reflected image of the star from obscuring the dimly seen horizon. This experiment suggests that it should be possible to design a sextant explicitly for night observations and that such an instrument should make it possible to obtain useful fixes at any time during a night when the visibility is good. For night use the sextant should have index and horizon mirrors of small size since only the brighter stars need be used (to simplify identification and avoid confusion with neighbouring stars). The horizon glass and the frame which surrounds it should be eliminated as they have no useful function, serving only to obscure the horizon. With these unnecessary optical parts removed, a night monocular with large diameter objective may be mounted, the size of the objective being limited only by weight of the finished instrument. A suitably filtered red light should be provided for reading the arc.

The σ values for observations at t^*'s ranging up to about $0'.55$ average at about $\sigma = 0'.58$, higher than the $\sigma = 0'.4$ characteristic of Sun sights. There are two reasons for the somewhat poorer reproducibility of the star sights: first, in twilight the horizon is not so well defined as in daylight and, second, the star image seen through the sextant is not a point but, rather, a blurred disc whose diameter is given in Table III. Thus, its position is not so well defined as the limb of the Sun where a sharper image is expected on the basis of diffraction theory and because of the fact that the eye can average around the periphery of the Sun in judging its limb.

At small t^*'s, where the horizon is well lighted, no better accuracy is attained by using either a larger magnification than normal (8×) or a telescope with a large objective glass (the 3×). This is in accord with the results for the reproducibility of the daylight observations. With all of the telescopes it is found that points rejected on the basis of Chauvenet's criterion occur at random with a frequency of about one in twenty. These 'rejects' are more or less evenly distributed along t^*: there is no tendency for the frequency of rejects to increase as t^* increases.

In summary, the results of the twilight experiments show that the reproducibility of star sights is slightly lower than that of Sun sights with a horizon of comparable quality, that with ordinary equipment star sights can be made reliably until t^* reaches about 0·55, and that when the conventional sextant telescope is replaced by an 8×30 monocular, star sights of useful accuracy can be made at any time during a clear night. A special sextant could easily be designed for this work. The influence of moonlight on star sights is discussed later.

5. SYSTEMATIC ERRORS. The systematic errors (lack of *precision* as contrasted with lack of accuracy due to random error) present in a measurement of the altitude of a celestial body by marine sextant can be divided into three classes:

(i) *Instrumental Errors*. This includes any error in the index correction of the sextant and errors in graduation, centring or optical alignment. The latter three can be detected by means of measurements on an optical bench; they may either be eliminated by appropriate adjustment of the instrument or a correction table may be prepared. Only the determination of the index correction remains a serious problem in the actual use of the instrument.

(ii) *Errors due to Atmospheric Refraction*. All sextant observations require refraction corrections and these are accurate only in so far as the density distribution of the atmosphere at the time of observations is the same as that assumed in the preparation of the correction tables. Refraction influences the dip of the horizon as well as the apparent altitude of the body observed.

(iii) *Observational Errors*. This includes the psychological-physiological factors involved in judging tangency to the horizon, sometimes called the 'personal equation', and the question of how the horizon may be resolved or distinguished when seen through different optical systems, under different types of illumination or varying degrees of obscuration.

It would be very difficult and costly to simulate the sources of systematic error in a laboratory. Consequently, a large number of observations must be made under a variety of carefully observed conditions and an attempt made to separate the magnitudes of the various systematic errors from these.

The problem of determining the index error has already been discussed. The other instrumental errors can be determined by laboratory measurement and are not of further concern here; presumably a sextant to be used in careful navigation will be in proper adjustment and the corrections for centring error provided by the maker will be applied.

Daylight Observations. The factors which influence the systematic error in the daylight observations are expressed in much the same way as for the random error. The average magnitudes of the Δh observed under various conditions are recorded in Table V. The limits of error given are the standard deviations for individual observations divided by the square root of the number of observations which, in each case, is at least nine. Where several strings of observations are included in the average, the greatest range of h values is also included and the value of $\Delta \bar{h}$ is the average of the Δh's taken regardless of sign.

Under 'ideal conditions' are included all observations made under ideal observing conditions when the dip and refraction could be expected to be normal, i.e. during the spring and summer when the air temperature was well above water temperature. Under these conditions results of very high accuracy are consistently attained, essentially to the limit to which the sextant can be read. Results are equally satisfactory with the 2½× and 8× telescopes showing that high magnification is not required in order to attain the inherent accuracy of a sextant graduated to 10 seconds of arc.

With the observer in motion in a small boat there is, on the average, a slight tendency in the observations for Δh to be too large. This might be ascribed to some uncertainty in the dip due to waves. However, when these data were taken, the waves were small enough compared to the length of the boat so that its centre could be considered to be at a nearly constant elevation. This is also indicated by the nearly constant standard deviation. At the horizon, only the average of the crests of the waves are seen and this should have the effect of making the apparent horizon too high and Δh too small, the opposite of what is observed. It is difficult to swing the arc properly when making observations from a small boat in rough water because of the yawing; this would tend to make the average of the observed altitudes too great and is probably the cause of the positive Δh's observed.

A poor horizon can arise in two different ways: with the visibility sufficient to allow the true horizon to be seen, the contrast between sky and water may be poor, or, alternatively, the visibility may be restricted by haze or fog making the horizon indistinct. In the first case, there should be no error in the dip (assuming normal refraction) but the measured h may be in error due to the difficulty of establishing tangency between Sun and horizon. In the second case there is an additional source of error introduced in determining the dip short of the true horizon. The distance to the apparent horizon must be accurately known or the dip will be considerably in error. For example, with an eye height of

10 ft. the dip for an apparent horizon distance 1·0 miles is 6′·1 and for a distance of 1·1 miles, 5′·6. Without buoys or other range marks available, it is nearly impossible to judge the distance to the apparent horizon with the requisite accuracy. The data in Table V show that with full visibility but an indistinct horizon, errors of the order of one minute of arc are to be expected. But when the visibility is restricted and the horizon poor (due, for example, to a lack of small waves to provide contrast between water and sky) much greater errors are encountered. In the first two sets of data listed under 'Restricted visibility, poor horizon' altitudes were corrected for dip on the basis of the estimated distance to the apparent horizon and Table 22 in *Bowditch*[9]. Distance to the apparent horizon was estimated on the basis of the visibility of the channel buoys leading into Guilford Harbor and should be accurate to within two or three-tenths of a mile. Nevertheless, the uncertainty in the estimated distance to the apparent horizon makes an important contribution to the observed Δh values.

TABLE V. SYSTEMATIC ERRORS IN DAYLIGHT OBSERVATIONS

Conditions of observations	Telescope	Δh	Δh_{max}	Δh_{min}
Ideal	2½× or 8×	0′·11 ±0·12	+0′·08	−0′·15
Cloud Cover only	2½×	+0·5 ±0·1		
Motion Only				
Yawing, at anchor	2½×	+0·14 ±0·10		
Reaching, Force 3 wind	2½×	+0·7 ±0·15		
	8×	0·7 ±0·2	+0·9	−0·4
Horizon Only				
Full visibility, poor 'horizon'	2½×	0·7 ±0·12	+0·9	−0·6
Restricted visibility, poor 'horizon'	2½×	5·7 ±1	+6·1	−6·9
Restricted visibility, poor 'horizon'	8×	7·4 ±1	+9·0	−5·9
Restricted visibility*, poor 'horizon'	2½×	−0·7 ±0·2		
Restricted visibility*, poor 'horizon'	8×	+1·3 ±0·2		
Abnormal Refraction*	2½×	1·0 ±0·1	+2·2	−0·1

*Δh based on dip determined by direct observation.

In the second set of data listed under 'Restricted Visibility', the dip meter was used to determine the apparent horizon. It is seen that the magnitude of Δh is greatly reduced when the dip is determined directly. If, in this particular set of experiments, the tabulated dip based on the estimated visibility had been used, Δh would have been larger by 3′·6. If this were added to the tabulated Δh values, they would still be somewhat less than the Δh of the first set; this is because the definition of the horizon was somewhat better when the second set of data was taken than during most of the experiments which contributed to the averages of the first set. It may be concluded that under conditions of reduced visibility where the distance to the apparent horizon can be estimated to within a few tenths of a mile, the uncertainty of this distance and the indistinctness of the horizon are about equally important sources of error. If it were not for the former effect, Sun sights would be of considerable practical utility in navigating vessels in pilotage waters during reduced visibility.

When making observations of the Sun with the horizon obscured by haze, there is a quite regular and systematic difference in the altitudes determined with the 2½× and 8× telescopes. This is shown by the data in Table V and also, for one set of observations, in Fig. 4. A smaller altitude is always measured with the 2½× telescope under these conditions.

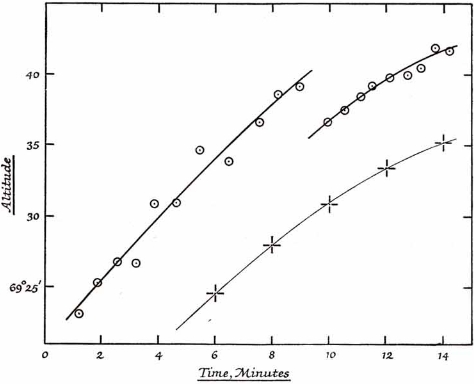

FIG. 4. Altitudes of the Sun taken, on the left, with the 8× and, on the right, with the 2½× telescopes on the Hughes sextant under conditions of restricted visibility and a poor horizon. A significant difference in the Δh obtained with the two telescopes is evident.

To reveal the effect reliably it is necessary for one set of observations to follow immediately on the other as, under conditions of restricted visibility, small changes in the visibility or other conditions produced marked changes in h and so may obscure the influence of the telescope itself. No explanation of this effect has been discovered as yet.

After a poorly defined horizon, abnormal atmospheric refraction is the next most important source of systematic error in altitude measurement by the marine sextant. From early March through mid-summer it was found that observations made under good conditions during the mid-day hours have a very small Δh. During this period the air temperature is well above the water temperature. In Long Island Sound the water becomes quite warm during the summer so that in late summer and autumn, air temperatures nearly the same or below water temperature are common. As soon as this situation obtains, appreciable systematic errors in mid-day observations are found. Abnormal atmospheric refraction will, in general, alter both the dip of the horizon and the apparent altitude of the body observed. In order to separate out these two contributions to Δh, the dip meter was used whenever the occurrence of abnormal refraction was suspected. The values of $\overline{\Delta h}$, $\Delta h_{\max}$, and $\Delta h_{\min}$ listed in Table V under 'Abnormal Refraction' are the departures of the observed and calculated altitudes when the observations are corrected for the measured rather than the tabulated dip. With the air colder than the water the observed dip is always found to be numerically greater than tabulated. In observations made during August, September and October the average departure of the observed from the tabulated dip was $2'.4$ and the greatest observed departure was $6'.6$ which, for the eye heights used, means that the horizon was depressed by the atmospheric refraction rather than raised as in the usual case. As shown by the last entries in Table V, the total altitude error in the presence of abnormal refraction is not entirely due to abnormal dip and, as the data were all taken under good conditions of observation which earlier in the year had been demonstrated to lead to very small altitude errors, it is evident that the abnormal atmospheric refraction has also significantly altered the apparent altitudes. This result was unexpected because the observed altitudes were all 30° or greater so that the refraction corrections should be small.

With the air temperature less than that of the water, air near the surface is heated and its density reduced. This heated layer of air is unstable relative to the colder and denser air immediately above it; this instability can be observed visually by examining the horizon with a moderately powerful telescope: a continual 'boiling' effect is seen. The velocity of light is faster than normal in the heated layer just above the water and so rays from the horizon are refracted upwards making the apparent dip of the horizon greater than normal, as is observed. The origin of the change in the apparent altitude is not so easily explained. For parallel layers in the atmosphere the magnitude of the celestial refraction, Ψ, is[10]

$$\Psi = 16''.17\frac{T}{B}\tan\zeta - 0''.07\tan\zeta\sec\zeta$$

where ζ is the apparent zenith distance, B is the barometric pressure in millibars and T the absolute centigrade temperature. Unless the apparent altitude be very small, the second term, the correction for the spherical curvature of the layers of the atmosphere, is altogether negligible at the level of precision attainable with the marine sextant. For an apparent altitude of 30°, tan $\zeta = 1{\cdot}7$ and $\Psi = 1'\!.6$. To account for the Δh in Table V listed under 'Abnormal Refraction', Ψ would have to be more than doubled in magnitude; this is not possible for any reasonable values of B and T. The source of the abnormal celestial refraction must be, then, in a departure from the conditions assumed in the deviation of the standard refraction equation above. One hypothesis is that the layers of constant density in the atmosphere are not horizontal near the place of observation. If, throughout the atmosphere, the surfaces of constant density are tilted, then the above equation gives the correct celestial refraction provided that Ψ is measured to the surfaces of constant density rather than to the horizontal. With the Sun at an altitude of about 35°, these surfaces would have to be inclined at about 15° in the direction shown in Fig. 5

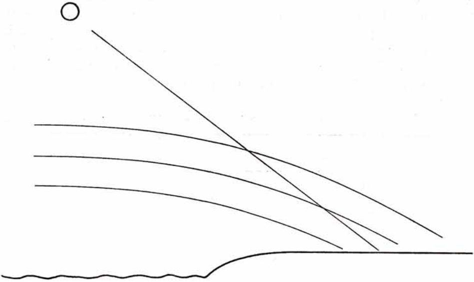

FIG. 5. Hypothetical curves of constant density in the atmosphere to explain abnormal refraction in Sun sights made from the shore. The sea is to the left, the warmer land to the right.

to account for a Δh of $1'\!.8$. Such a great inclination of the constant density surfaces could never obtain over a large region of the atmosphere but it is hypothesized that they might result from local heating of the air over the shore during mid-day. When the inclination of the constant density surfaces is downwards looking towards the body observed, as might be the case when a cold front is advancing from the land over the water, then a negative Δh is expected. However, the magnitude of this effect on Δh would have to be small as the apparent zenith distance is being decreased.

If the above explanation is correct, it would indicate that the error in

h due to refraction effects as observed from the shore is not representative of the errors from this effect which might be encountered at sea. To settle this question data would have to be collected from a ship in a known location under reasonably well-known atmospheric conditions. There is no doubt, however, that abnormal dip will be encountered at sea and it is thought that the dip data reported here are probably representative of the magnitude of the departure from normal dip which may be expected. Abnormal dip is frequently, but not necessarily, accompanied by inferior mirage; this can be detected even when very weak by examination of a distant object through a powerful telescope. Under these conditions accurate Sun lines can only be determined when a dip meter is used in conjunction with the sextant.

Evening Observations. For some 300 star altitudes measured from March through October Δh was never found to be zero. In a few cases Δh was negative but its average was about $+ 3'$ and the greatest positive value, $+ 7'.5$. There is a tendency for Δh to increase as twilight deepens as shown in Table VI. In this table the altitudes for large t^* were taken with the 8× telescope but there is no reason to expect that this, by itself, would influence Δh as at small t^*, when the horizon can be seen well with either the 8× or the 2½× telescope, no systematic difference exists between altitudes taken with the two of them.

TABLE VI. VARIATION OF Δh WITH t^* DURING EVENING TWILIGHT

Arcturus		Altair	
t^*	Δh	t^*	Δh
0·21	$+ 5'.7$	0·55	$+ 6'.8$
0·26	6·2	0·71	7·5
0·31	6·9	0·86	7·6
0·36	6·3	1·01	8·13
0·42	5·5	1·17	10·3
0·47	5·5		
0·52	5·4		

The positive Δh found for star sights must be due to some combination of departure from normal dip and refraction plus any error due to difficulty judging tangency to the dimly illuminated horizon. At small t^* dip could be determined directly but the light gathering power of the dip meter telescope was not sufficient to allow use of this instrument at large t^*. Table VII shows the altitude errors observed on one occasion when the dip was determined directly. In this table D is the measured dip, ΔD the observed departure of the dip from its tabulated value for the eye height of the observer, and $\Delta h - \Delta D$ is the residual error in h after correction for the measured dip. Since the true dip is determined directly, the remaining error, $\Delta h - \Delta D$, must be due to either abnormal celestial refraction or erroneous judgment of tangency, or both. If the residual error were due to refraction, it should be considerably larger for Saturn than for Altair due to the lower altitude of the former.

TABLE VII. DIP AND ALTITUDE OBSERVATIONS, EVENING TWILIGHT

Body	D	ΔD	h	Δh	$\Delta h - \Delta D$
Saturn	3′·6	+1′·0	25°	+4′·2	+3′·2
Altair	3·6	+1·0	57	+4·3	+3·3

Evidently abnormal refraction does not make a significant contribution to Δh in this case and the residual error must be due to visual difficulty in judging the horizon. The increase in Δh with t^* agrees with this idea as does the failure to find any good correlation between Δh and the difference between air and water temperature. Another effect of horizon illumination on Δh is shown in Table VIII, which shows the results of three series of observations taken in sequence during an evening when the horizon was illuminated by a nearly full Moon.

TABLE VIII. EVENING OBSERVATIONS UNDER MOONLIGHT

Body	Azimuth relative to Moon	h	$\Delta h - \Delta D$
Saturn	18°	17°	+3′·1
Altair	10	50	+3·0
Moon	0	19	+4·0

As in Table VII, the residual error for Saturn and Altair is about the same; these bodies were observed on a part of the horizon differing sufficiently in azimuth from the Moon's azimuth so as not to be illuminated directly by reflected moonlight. The Moon's altitude, however, was observed in the centre of a path of brightly reflected moonlight. This had the effect of lowering the apparent horizon by about 1′·0.

On the basis of the above results it would be expected that the effect of Δh on a fix obtained at sea during evening twilight could be eliminated by making a balanced set of sights. It would be necessary to assume that the dip was the same in all directions about the ship. For the best accuracy the observations should be planned so that observations on reciprocal azimuths are made with equivalent conditions of horizon illumination.

6. DISCUSSION OF THE EXPERIMENTAL RESULTS. The primary object in making a comprehensive study of the sources of error in celestial navigation as practised with the marine sextant is to discover the relative

importance of all the sources of error which contribute to inaccuracy in the final fix. If, for example, abnormal celestial refraction were the principal source of error, there would be little point in devoting time to the improvement of the sextant itself. In fact, it is found that under very good or ideal conditions of observation during daylight, altitudes can be measured to an accuracy comparable with that of which the instrument itself is capable. Under these conditions, refinements in the sextant—improved telescopes and smaller graduation errors—would lead to improved accuracy in a line of position. However, even the ordinary instrument used in this work is capable, through a series of observations, of accuracy to within $0'.2$ under good daylight conditions and it is doubtful if the improvement attained with a better instrument would be worth the additional trouble and expense involved. Also, the improved accuracy would only be attained when fixing the position of a moored vessel because of the errors inherent in advancing a line of position.

Inaccuracy apart from errors due to the sextant develops when the horizon is poorly defined, the visibility restricted, or atmospheric refraction abnormal: the standard deviation of the observations is increased but there is also a systematic error introduced significantly greater than the random error. Thus improved fixes will result from the use of a balanced set of observations on reciprocal azimuths and any improvements in sextant design which increases the time over which star sights can be made will be a significant development. Such improvements should include making the stars visible at smaller t^* values while the horizon is still well illuminated, and increasing the visibility of the horizon to permit observations to continue to large t^*. The latter is easier to realize and, as discussed above, an instrument could be designed which would permit star sights to be made throughout the night whenever the weather is reasonably clear. The characteristics of a sextant for this purpose have already been enumerated. The great advantage of night observations is that well after sunset relatively stable atmospheric conditions have their best chance to develop. The error Δh on all azimuths about the ship will then be most nearly equal and so can be eliminated on comparing sights on reciprocal azimuths. The development of an instrument to make stars visible during daylight or early twilight will be a more difficult problem. High magnification telescopes will be required. It is doubtful whether the use of front surface mirrors, as proposed by some authors[6], will be of any advantage; the problem is to separate sky light from star light, not make the overall illumination of the visual field brighter.

A very worthwhile improvement in sextant design would be a means of holding the index error constant. This error could then be determined accurately ashore and an important source of uncertainty in accurate measurements at sea would be eliminated. Most modern sextants are deficient in the rigidity of their horizon glass mounting.

More rapid or convenient methods of sight reduction are not of major importance to precision celestial navigation. The time required for making a good set of observations, plotting them and extracting the necessary almanac data is already a substantial part of the total time required to work the sights. It is almost as fast to solve the sine-cosine equation for altitude as it is to work altitudes from H.O. 214 by triple interpolation. If a specially prepared set of tables with the natural sines and cosines and their logarithms entered for each 0'.1 were available, it is believed that solution of the sine-cosine formula by logarithms would be faster than solution for h by H.O. 214. Direct calculation is to be preferred in careful work because of its inherently higher accuracy. No advantage in the use of the haversine method is evident. H.O. 214 suffices for azimuth determination if the dead reckoning position is reasonably good.

Developments which will improve the precision and accuracy of celestial navigation are worthy of pursuit in spite of the increasingly widespread use of long-range, radio navigation systems. By precision celestial navigation any ship, with simple and highly reliable equipment, can make an accurate absolute check on the accuracy of the radio navigation system. Presumably this would be done whenever conditions for observation were favourable; because it would be done less frequently than routine daily navigation, it would be worth the extra time and trouble required to get highly reliable results.

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