Average Semidiameter

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Define

a	semi-major axis of the orbit
e	orbital eccentricity
s	Sun's radius
ν	the true anomaly
P	the Earth's orbital period

then the semi-minor axis of the orbit is $b = a\sqrt{1-e^2}$

and the Sun-Earth distance is $r = \dfrac{a\left(1-e^2\right)}{1+e\cos\nu}$ which takes extreme values $r_{\max} = a\left(1+e\right)$, $r_{\min} = a\left(1-e\right)$.

From conservation of angular momentum or equivalently Kepler's second law it can be shown that $\dfrac{P}{2\pi a^2\sqrt{1-e^2}} r^2 d\nu = dt$.

In the approximation $\sin^{-1}\dfrac{s}{r} \approx \dfrac{s}{r}$ The time averaged semidiameter is

$$\frac{1}{P}\int_0^P \frac{s}{r} dt = \frac{s}{2\pi a^2\sqrt{1-e^2}} \int_0^{2\pi} r\, d\nu = \frac{s}{a} .$$

The average of the maximum and minimum semidiameters is $\dfrac{1}{2}\left(\dfrac{s}{r_{\max}} + \dfrac{s}{r_{\min}}\right) = \dfrac{s}{a\left(1-e^2\right)}$.

Proportion of time with r > a

Use $r = \dfrac{a\left(1-e^2\right)}{1-e\cos\theta}$ where $\theta = 0$ corresponds to aphelion Then $r = a$ when $\cos\theta = e$. The area swept out while $r \geq a$ is $2\displaystyle\int_0^{\cos^{-1}e} \frac{r^2}{2} d\theta = -2\int_1^e \left(\frac{a\left(1-e^2\right)}{1-eu}\right)^2 \frac{1}{2\sqrt{1-u^2}} du = a^2\sqrt{1-e^2}\left(e + \frac{\pi}{2}\right)$

where the substitution $u = \cos\theta$ has been made. The total area of the ellipse is $\pi ab = \pi a^2\sqrt{1-e^2}$ so the proportion is $\dfrac{1}{2} + \dfrac{e}{\pi}$.