usually the sea horizon (VISIBLE HORIZON) and for an aviator the horizon is usually an artificial horizon (SENSIBLE HORIZON) in a bubble sextant. Figure 9.2 shows the Visible Horizon, VV', and the sensible horizon, SS', for an observer at the point Z whose height above the surface of the Earth is h. The angle SZV is known as the DIP of the sea horizon.

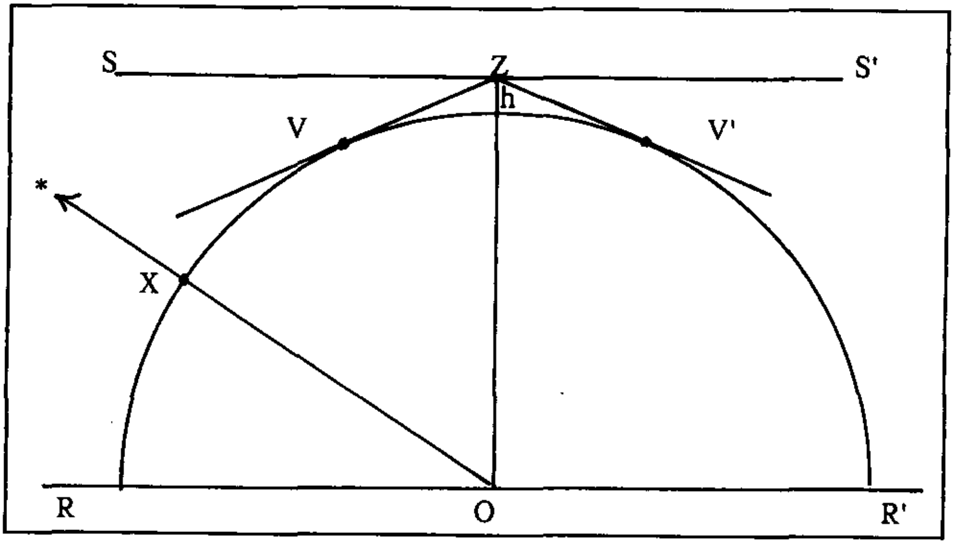

FIGURE 9.2: OBSERVER'S VISIBLE AND SENSIBLE HORIZONS.

The plane through the centre of the Earth which is parallel to the plane which is the sensible horizon is the RATIONAL HORIZON. (RR' in figure 9.2). After the altitude has been read from the sextant it is corrected to give the altitude of the centre of the body from the centre of the Earth above the rational horizon. The ALTITUDE of the body * is then the angle RO* in figure 9.2. The point X is the geographical position of the the body *. Having found the altitude of an astronomical body, *, we subtract this value from 90° to give us the ZENITH DISTANCE of * which is the angle ZO* in figure 9.2.

9.3 THE COSINE FORMULA.

We have seen that when an astronomical observation is taken from the surface of the Earth the observer corrects the reading given by the sextant to give the angle subtended at the centre of the Earth between the normal to the surface in the position of the observer and the line from the observed body to the centre of the Earth. In the case of the sphere these lines are along the radii meeting at the centre of the sphere and both are normal to the surface of the sphere. To assume that the earth is a sphere, therefore, makes the analysis of the computation fairly straightforward. To make a rigorous analysis, however, and allow for the ellipticity of the Earth we must take into account that the angle observed is no longer the angle between two normals at these positions but the angle between the normal to the surface at the position of the observer and the line from the body to the centre of the Earth. In general the normal to the surface of the ellipsoid does not pass through the centre of the ellipsoid so that these lines are not quite in the same plane, but, using the approximation based on the vast difference between the magnitude of the Earth's radius and the distance between the Earth and any astronomical body, the two lines may be considered to meet at a point at the centre of the earth just the same. It might seem that, in the case of an ellipsoid, the methods of computing the coordinates of the observer's position using spherical trigonometry no longer apply but, using the methods of differential geometry, we can show that, provided that we make these simple assumptions about the intersection of these lines, the observer's position circle can still be described using the cosine formula.

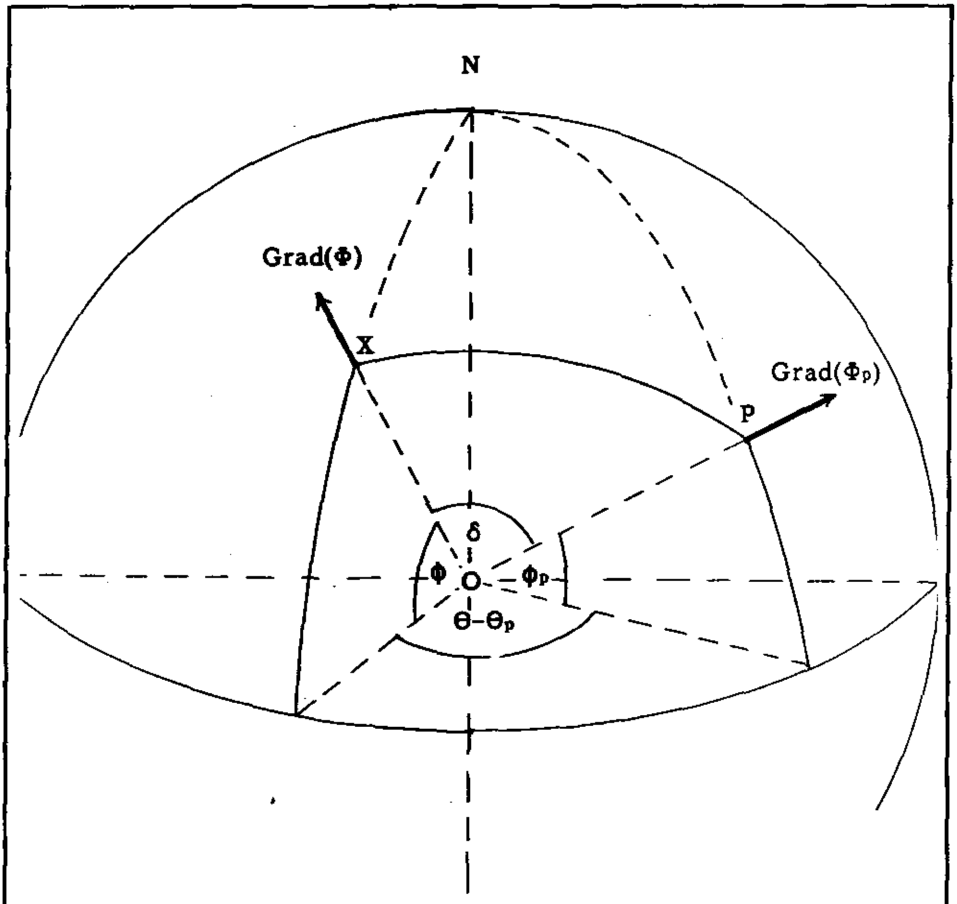

FIGURE 9.3

In general, in Cartesian coordinates, a closed surface may be expressed in the implicit form $\Phi(x,y,z) = 0$ which in the case of the sphere, may be written

$$\Phi(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0 \qquad (9.1)$$

where a is the radius of the sphere, the origin of coordinates is at the centre, O, of the sphere and, for the Earth, the Z axis lies along the axis of revolution and the Greenwich meridian lies in the plane $y=0$.

The normal to the surface $\Phi(x,y,x) = 0$ is in the direction of the vector **grad(Φ)** where

$$\mathbf{grad}(\Phi) = [\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}] .$$

For the sphere defined by equation (9.1) we find that

$$\mathbf{grad}(\Phi) = [2x,2y,2z].$$

If the observer's position, X, is at the point defined by coordinates (x,y,z) and the geographical position is at the point P with coordinates (x_P,y_P,z_P) then the angle POX ($=\delta$), between the normal to surface at X and the normal to the surface at P is given in vector notation by

$$\cos\delta = \frac{\mathbf{grad}\ \Phi \cdot \mathbf{grad}\ \Phi_P}{|\mathbf{grad}\ \Phi| \cdot |\mathbf{grad}\ \Phi_P|} \qquad (9.2)$$

The angle δ is the ZENITH DISTANCE. See Figure 9.3.

If we transform to spherical coordinates using the coordinate transformations

$$x = \cos\phi\cos\theta \qquad y = \cos\phi\sin\theta \qquad z = \sin\phi$$

where ϕ is the latitude and Θ is the longitude, then some simple algebraic manipulation will show that equation (9.2) reduces to

$$\cos\delta = \cos\phi\cos\phi_P\cos(\Theta - \Theta_P) + \sin\phi\sin\phi_P \qquad (9.3)$$

which is the familiar spherical cosine formula.

The angle $(\Theta - \Theta_P)$ is what we know as the LOCAL HOUR ANGLE.

If, instead, we consider that the Earth is better approximated by an ellipsoid of revolution, then the surface can be represented by the equation

$$\Psi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0 \qquad (9.4)$$

where a is the length of the semi major axis of the meridian ellipse and b is the semi minor axis. On the surface of the ellipsoid we find

$$\mathbf{grad}(\Psi) = [\frac{2x}{a^2}, \frac{2y}{a^2}, \frac{2z}{b^2}]$$

and $$|\mathrm{grad}(\Psi)| = \sqrt{\left[\frac{4x^2}{a^2} + \frac{4y^2}{a^2} + \frac{4z^2}{b^2}\right]}$$

The observed body is mapped onto the surface of a sphere (the celestial sphere) and along the line of the vector $\mathbf{r}_p = [x_p, y_p, z_p]$ for which the unit vector is $\hat{\mathbf{r}}_p = [\xi_p, \eta_p, \zeta_p]$ where $\xi_p^2 + \eta_p^2 + \zeta_p^2 = 1$.
In spherical coordinates $\hat{\mathbf{r}}_p$ is defined by the vector

$$\hat{\mathbf{r}}_p = [\cos\phi_p \cos\Theta_p,\ \cos\phi_p \sin\phi_p,\ \sin\phi_p]$$

where ϕ_p is the declination of the observed body and Θ_p is the longitude of its geographical position. The angle δ between the vectors $\hat{\mathbf{r}}_p$ and $\mathrm{grad}(\Psi)$ which is the angle which gives us the zenith distance is given by $\cos\delta = \dfrac{\hat{\mathbf{r}}_p \cdot \mathrm{grad}\,\Psi}{|\mathrm{grad}\,\Psi|}$.

Some manipulation will then yield

$$[(1-e^2)^3(x\xi_p + y\eta_p) + z\zeta_p]^2 - [(1-e^2)^2(x^2 + y^2) + z^2]\cos^2\delta = 0 \qquad (9.5)$$

If we express equation (9.5) in spherical coordinates with
$r_p = a\sqrt{\left[\dfrac{1-e^2}{1 - e^2\cos^2\phi_p}\right]}$ and rearrange we find

$$[(1-e^2)^2\cos\phi \cos\phi_p \cos(\Theta-\Theta_p) + \sin\phi \sin\phi_p]^2 - [(1-e^2)^2\cos^2\phi + \sin^2\phi]\cos^2\delta = 0$$

If we now make the substitution $\tan\phi = (1-e^2)\tan\psi$ where ψ is the geodetic latitude, then

$$[\cot\psi \cos\phi_p \cos(\Theta-\Theta_p) + \sin\phi_p]^2 - \mathrm{cosec}^2\psi \cos^2\delta = 0 \qquad (9.6)$$

The expression in equation (9.6) can be factorised so that we will find

$$\boxed{\cos\delta = \cos\psi \cos\phi_p \cos(\Theta-\Theta_p) + \sin\psi \sin\phi_p} \qquad (9.7)$$

which is the same form as equation (9.3) and is, once again, the spherical cosine formula.

9.4 FINDING THE OBSERVED POSITION.

A second observation, taken simultaneously of a different body with geographical position at a point Q whose latitude, ϕ_q, is given by the declination of the body and whose longitude, Θ_q, is obtained from the