In this case, $t_{0.025,3} = 3.183$ and the 95% probable error for the angular sum is

$$\Sigma_{95\%} = 3.183(24.7'') = \pm 78.6''$$

Thus, the traverse angles are well within the range of allowable error. We cannot reject the null hypothesis that the error in the angles is not statistically equal to zero. Thus, the survey meets the minimum level of angular closure at a 95% probable error. However, it must be remembered that because of the possibility of Type II errors, we can only state that there is no statistical reason to believe that there might be a blunder in the angle observations.

Example 7.9 presents another question for the statistician or surveyor. That is, should a surveyor allow a field crew to have this large an angular misclosure in the traverse? Statistically, the answer would seem to be yes, but recall that the target- and instrument-centering errors affect angle observations only if the instrument and targets are reset after each observation. Since this is seldom done in practice, these two errors should not be included in the summation of the angles. Instead, the allowable angular misclosure should be based solely on the pointing and reading errors. For example, if the angles were observed with a total station having a DIN 18723 standard of $\pm 1''$, by Equation (7.7) the pointing and reading error for each angle would be

$$\sigma_{\alpha_{pr}} = \pm \frac{2 \times 1''}{\sqrt{2}} = \pm 1.4''$$

By Equation (6.19), the error in the summation of the five angles would be $\pm 1.4''\sqrt{5} = \pm 3.2''$. Using the same critical t value of 3.183, the allowable error in the angular misclosure should only be $3.183 \times 3.2'' = \pm 10''$. If this instrument were used in Example 7.9, the field-observed angular closure of $30''$ would be unacceptable and would warrant reobservation of some or all of the angles.

As stated in Sections 7.6 and 7.7, the angular misclosure of $78.6''$ computed in Example 7.9 will only be noticed when the target and instrument were reset on a survey. This is most likely to occur in a resurvey, during which the centering errors of the target and instrument from the original survey will be present in the *record* directions. Thus, record azimuths or bearings could disagree from those determined in the resurvey by this amount, assuming that the equipment used in the resurvey is comparable or of higher quality than that used in the original survey.

7.11 ERRORS IN ASTRONOMICAL OBSERVATIONS FOR AZIMUTH

The total error in an azimuth determined from astronomical observations depends on errors from several sources, including those in timing, the observer's latitude and longitude, the celestial object's position at observation time, timing accuracy, instrument optics, atmospheric conditions, and others, as identified in Section 7.2. The error in astronomical observations can be estimated by analyzing the hour-angle formula, which is

$$z = \tan^{-1} \frac{\sin t}{\cos\phi \tan\delta - \sin\phi \cos t} \tag{7.25}$$

In Equation (7.25), z is the angle used to compute the azimuth of the celestial object at the time of the observation, t the *t angle* of the *PZS* triangle at the time of observation, ϕ the observer's latitude, and δ the object's declination at the time of the observation.

The t angle is a function of the *local hour angle* (LHA) of the sun or star at the time of observation. That is, when the LHA $<180^\circ$, $t = \text{LHA}$; otherwise, $t = 360^\circ - \text{LHA}$. Furthermore, LHA is a function of the *Greenwich hour angle* (GHA) of the celestial body and the observer's longitude; that is,

$$\text{LHA} = \text{GHA} + \lambda \tag{7.26}$$

where λ is the observer's longitude, considered positive for eastern longitude and negative for western longitude. The GHA increases approximately 15° per hour of time, and thus an estimate of the error in the GHA is approximately

$$\sigma_t = 15^\circ \sigma_T \tag{7.27}$$

where σ_T is the estimated error in time (in hours). Similarly, by using the declination at 0^h and 24^h, the amount of change in declination per second can be derived and thus the estimated error in the hour angle determined.

Using Equation (6.16), the error in a star's azimuth is estimated by taking the partial derivative of Equation (7.25) with respect to t, δ, ϕ, and λ. To do this, simplify Equation (7.25) by letting

$$F = \cos\phi \, \tan\delta - \sin\phi \, \cos t \tag{7.28a}$$

and

$$u = \sin t \times F^{-1} \tag{7.28b}$$

Substituting Equations (7.27), Equation (7.25) is rewritten as

$$z = \tan^{-1} \frac{\sin t}{F} = \tan^{-1} u \tag{7.29}$$

From calculus it is known that

$$\frac{d \tan^{-1} u}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$$

Applying this fundamental relation to Equation (7.28) and letting G represent GHA yields

$$\frac{\partial z}{\partial G} = \frac{1}{1 + \left[\sin(G - \lambda)/F\right]^2} \frac{du}{dG} = \frac{F^2}{F^2 + \sin^2(G - \lambda)} \frac{du}{dG} \tag{7.30}$$

Now du/dG is

$$\begin{aligned}\frac{du}{dG} &= \frac{\cos(G - \lambda)}{F} - \frac{\sin(G - \lambda)}{F^2} \sin\phi \sin(G - \lambda) \\ &= \frac{\cos(G - \lambda)}{F} - \frac{\sin^2(G - \lambda)\sin\phi}{F^2}\end{aligned}$$

and thus

$$\frac{du}{dG} = \frac{F\cos(G - \lambda) - \sin^2(G - \lambda)\sin\phi}{F^2} \tag{7.31}$$

Substituting Equation (7.30) into Equation (7.29) and substituting in t for $G - \lambda$ yields

$$\frac{\partial z}{\partial G} = \frac{F\cos t - \sin^2 t \sin\phi}{F^2 + \sin^2 t} \tag{7.32}$$

In a similar fashion, the following partial derivatives are developed from Equation (7.25):

$$\frac{dz}{d\delta} = -\frac{\sin t \cos\phi}{\cos^2\delta\left(F^2 + \sin^2 t\right)} \tag{7.33}$$

$$\frac{\partial z}{\partial \phi} = \frac{\sin t \cos t \cos\phi + \sin t \sin\phi \tan\delta}{F^2 + \sin^2 t} \tag{7.34}$$

$$\frac{\partial z}{\partial \lambda} = \frac{\sin^2 t \sin\phi - F\cos t}{F^2 + \sin^2 t} \tag{7.35}$$

In Equation (7.35), t is the t angle of the PZS triangle, z the celestial object's azimuth, δ the celestial object's declination, ϕ the observer's latitude, λ the observer's longitude, and $F = \cos\phi\tan\delta - \sin\phi\cos t$.

If the horizontal angle, H, is the observed angle to the right from the line to the celestial body, the equation for a line's azimuth is

$$Az = z + 360^\circ - H \tag{7.36}$$

Therefore, the error contributions from the horizontal angle observation must be included in computing the overall error in the azimuth. Since the distance to the star is considered infinite, the estimated contribution to the angular error due to the instrument-centering error can be determined with a formula similar to that for the target-centering error with one pointing. That is,

$$\sigma_{\alpha_i} = \frac{\sigma_i}{D} \tag{7.37}$$

where σ_i is the centering error in the instrument, and D is the length of the azimuth line in the same units. Note that the results of Equation (7.37) are in radian units and must be multiplied by ρ to yield a value in arc seconds.

Example 7.10 Using Equation (7.25), the azimuth to Polaris was found to be $0^\circ 01'31.9''$. The observation time was 1:00:00 UTC with an estimated error of $\sigma_T = \pm 0.5^s$. The Greenwich hour angles to the star at 0^h and 24^h UTC were $243^\circ 27'05.0''$ and $244^\circ 25'50.0''$, respectively. The LHA at the time of the observation was $181^\circ 27'40.4''$. The declinations at 0^h and 24^h were $89^\circ 13'38.18''$ and $89^\circ 13'38.16''$, respectively. At the time of observation, the declination was $89^\circ 13'38.18''$. The clockwise horizontal angle observed from the backsight to a target 450.00 ft was $221^\circ 25'55.9''$. The observer's latitude and longitude were scaled from a map as $40^\circ 13'54''$N and $77^\circ 01'51.5''$W, respectively with estimated errors of $\pm 1''$. The vertical angle to the star was $39^\circ 27'33.1''$. The observer's estimated errors in reading and pointing are $\pm 1''$ and $\pm 1.5''$, respectively, and the instrument was leveled to within 0.3 of a division with a bubble sensitivity of $25''$/div. The estimated error in instrument and target centering is ± 0.003 ft. What are the azimuth of the line and its estimated error? What is the error at the 95% level of confidence?

SOLUTION The azimuth of the line is $\text{Az} = 0^\circ 01'31.9'' + 360 - 221^\circ 25'55.9'' = 138^\circ 35'36''$. Using the Greenwich hour angles at 0^h and 24^h, an error of 0.5^s time will result in an estimated error in the GHA of

$$\pm \frac{360^\circ + \left(244^\circ 25'50.0'' - 243^\circ 27'05.0''\right)}{24^h \times 3600^{s/h}} 0.05^s = \pm 7.52''$$

Since $t = 360^\circ - \text{LHA} = 178^\circ 32'19.6''$, F in Equations (7.29) through (7.34) is

$$\begin{aligned} F &= \cos(40^\circ 13'54'')\tan(89^\circ 13'38.18'') - \sin(40^\circ 13'54'')\cos(178^\circ 32'19.6'') \\ &= 57.249 \end{aligned}$$

The error in the observed azimuth can be estimated by computing the individual error terms as follows:

(a) From Equation (7.32) the error with respect to the GHA, G, is

$$\begin{aligned} \frac{\partial z}{\partial G}\sigma_G &= \frac{57.249\cos(178^\circ 32'19.6'') - \sin^2(178^\circ 32'19.6'')\sin(40^\circ 13'54'')}{57.249^2 + \sin^2(178^\circ 32'19.6'')} 7.52'' \\ &= \pm 0.13'' \end{aligned}$$

(b) By observing the change in declination, it is obvious that for this observation, the error in a time of 0.5^s is insignificant. In fact, for the entire day, the declination changes only $0.02''$. This situation is common for stars. However, the sun's declination may change from only a few seconds daily to more than 23 minutes per day, and thus for solar observations, this error term should not be ignored.

(c) From Equation (7.34) the error with respect to latitude, ϕ, is

$$\frac{\partial z}{\partial \phi}\sigma_\phi = \pm\frac{\sin t \cos t \cos\phi + \sin t \sin\phi \tan\delta}{F^2 + \sin^2 t}\sigma_\phi = \pm 0.0004''$$

(d) From Equation (7.35) the error with respect to longitude, λ, is

$$\pm\frac{\sin^2\left(178^\circ 32'19.6''\right)\sin\left(40^\circ 13'54''\right) - 57.249\cos\left(178^\circ 32'19.6''\right)}{57.249^2 + \sin^2\left(178^\circ 32'19.6''\right)} \times 1''$$
$$= \pm 0.02''$$

(e) The circles are read both when pointing on the star and on the azimuth mark. Thus, from Equation (7.2), the reading contribution to the estimated error in the azimuth is

$$\sigma_{\alpha_r} = \pm\sigma_r\sqrt{2} = \pm 1''\sqrt{2} = \pm 1.41''$$

(f) Using Equation (7.6), the estimated error in the azimuth due to pointing is

$$\sigma_{\alpha_p} = \pm\sigma_p\sqrt{2} = \pm 1.5''\sqrt{2} = \pm 2.12''$$

(g) From Equation (7.8), the estimated error in the azimuth due to target centering is

$$\sigma''_{\alpha_t} = \pm\frac{d}{D} = \pm\frac{0.003}{450}206{,}264.8''/\text{rad} = \pm 1.37''$$

(h) Using Equation (7.37), the estimated error in the azimuth due to instrument centering is

$$\sigma''_{\alpha_i} = \pm\frac{d}{D} = \pm\frac{0.003}{450}206{,}264.8''/\text{rad} = \pm 1.37''$$

(i) From Equation (7.23), the estimated error in the azimuth due to the leveling error is

$$\sigma_{\alpha_b} = \pm f_d\,\mu\tan v = \pm 0.3 \times 25''\tan(39^\circ 27'33.1'') = \pm 6.17''$$

Parts (a) through (i) are the errors for each individual error source. Using Equation (6.18), the estimated error in the azimuth observation is

$$\sigma_{\mathrm{AZ}} = \sqrt{(0.13'')^2 + (1.32'')^2 + (0.02'')^2 + (0.0004'')^2 + (2.12'')^2 + 2\,(1.37'')^2 + (6.17'')^2}$$
$$= \pm 7.0''$$

Using the appropriate *t-value* of $t_{0.025,1}$ from Table D.3, the 95% error is

$$\sigma_{AZ} = \pm 12.705(7.0'') = \pm 88.9''$$

Notice that in this problem, the largest error source in the azimuth error is caused by the instrument-leveling error.

7.12 ERRORS IN ELECTRONIC DISTANCE OBSERVATIONS

All electronically measured distance observations are subject to instrumental errors that manufacturers list as constant, a, and scalar, b, error. A typical specified accuracy is $\pm(a + b \text{ ppm})$. In this expression, a is generally in the range 1 to 10 mm, and b is a scalar error, which typically has the range 1 to 10 parts per million (ppm). Other errors involved in electronically measured distance observations stem from the target- and instrument-centering errors. Since in any survey involving several stations, these errors tend to be random, they should be combined using Equation (6.18). Thus, the estimated error in an EDM observed distance is

$$\sigma_D = \sqrt{\sigma_i^2 + \sigma_t^2 + a^2 + (D \times b \text{ ppm})^2} \tag{7.38}$$

where σ_D is the error in the observed distance D, σ_i the instrument-miscentering error, σ_t, the reflector-miscentering error, and a and b the instrument's specified accuracy parameters.

Example 7.11 A distance of 453.87 ft is observed using an EDM with a manufacturer's specified accuracy of $\pm$(3 mm + 3 ppm). The instrument is centered over the station with an estimated error of $\pm$0.003 ft, and the reflector, which is mounted on a handheld prism pole, is centered with an estimated error of $\pm$0.01 ft. What is the error in the observed distance? What is the E_{95} value?

SOLUTION Converting millimeters to feet using the survey foot[2] definition gives us

$$0.003\frac{39.37 \text{ in.}}{12 \text{ in.}} = 0.0098 \text{ ft}$$

[2]The survey foot definition is 1 meter = 39.37 inches, exactly.

$$\Sigma_{zz} = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} & \cdots & \frac{\partial Z_1}{\partial x_n} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial Z_m}{\partial x_1} & \frac{\partial Z_m}{\partial x_2} & \cdots & \frac{\partial Z_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{x_n}^2 \end{bmatrix}$$
$$\times \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_2}{\partial x_1} & \cdots & \frac{\partial Z_m}{\partial x_1} \\ \frac{\partial Z_1}{\partial x_2} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_m}{\partial x_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial Z_1}{\partial x_n} & \frac{\partial Z_2}{\partial x_n} & \cdots & \frac{\partial Z_m}{\partial x_n} \end{bmatrix} \tag{6.15}$$

If there is only one function Z, involving n unrelated quantities, $x_1, x_2, \ldots, x_n$, Equation (6.15) can be rewritten in algebraic form as

$$\sigma_Z = \sqrt{\left(\frac{\partial Z}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial Z}{\partial x_2}\sigma_{x_2}\right)^2 + \cdots + \left(\frac{\partial Z}{\partial x_n}\sigma_{x_n}\right)^2} \tag{6.16}$$

Equations (6.14), (6.15), and (6.16) express the *special law of propagation of variances* (SLOPOV). These equations govern the manner in which errors from statistically independent observations (i.e., $\sigma_{x_i x_j} = 0$) propagate in a function. In these equations, individual terms represent the individual contributions to the total error that occur as the result of observational errors in each independent variable. When the size of a function's estimated error is too large, inspection of these individual terms will indicate the largest contributors to the error. The most efficient method to reduce the overall error in the function is to examine ways to reduce the largest individual error terms in Equation (6.16) closely.

6.1.1 Generic Example

Let $A = B + C$, and assume that B and C are independently observed quantities. Note that $\partial A/\partial B = 1$ and $\partial A/\partial C = 1$. Substituting these into Equation (6.16) yields

$$\sigma_A = \sqrt{(1\sigma_B)^2 + (1\sigma_C)^2} \tag{6.17}$$