LADEE Moon Probe Sighting
Calculating the Observer's Location

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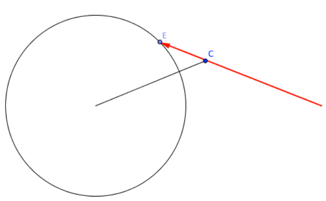
Problem Statement

On 6 September,2013 on schedule at 23:27 EDT the LADEE mission was launched from Wallops Island, Virginia and its ascent was visible from the East Coast of the US. Details of the launch trajectory can be found at http://www.orbital.com/NewsInfo/MissionUpdates/MinotaurV/files/LADEE-Mission-Overview.pdf I took the attached photographs from an undisclosed location. The first is included just for the wow factor and shows Stage 2 burn out followed by Stage 3 ignition along with the trails of some stars in Sagittarius. The second photo shows Stage 3 burn out against the background stars and also the rocket plume that was barely visible to the naked eye. The relatively bright star above the trail end is Fomalhaut. Information from the link above indicates that Stage 3 burn out was to occur at T +207.62s, Latitude 37.41, Longitude -69.43, Altitude 161.99km. From measurements of the photograph the last point visible of the rocket's trail appears close to R.A. 23h5m47.4s, Declination -31d46.1' (J2000.0). Where was the photo taken from? Robin Stuart

My original solution was as follows



I have my doubts on how accurate this approach is, but the idea is to find where the line from the direction defined by

R.A. $23^{h}05^{m}47.4^{s}$,

Declination -31° 46.1'at 2013 Sept 7, 03h 30m 28s Z

through the point

Latitude 37.41, Longitude -69.43, Altitude 161.99km intersects the Earth's surface. See figure.

With the given numbers, I come up with N 42.0477, W 71.1326

However, in the course of this write-up I found a few small errors. The corrected location, shown in Figure 1 as LADEE 2, is N 42.0983, W 71.1420.

Robin Stuart responded to the original posted solution and provided the following information:

2013 Sep 14, 07:49
Robert, The photos were taken from the North East corner of Lake Massapoag at 42d 06'39" N, 71d 10' 35". Your result of 42.0477 N, 71.1326 W lies 4.25nm in a direction roughly towards the rocket. Well done! My own calculation yielded 42.0346 N, 71.1172 W which is about 1 nm further out than yours but along the same direction. The difference between the true and computed positions translates into around a 10' error in declination so perhaps with a bit more care in the measurements one might be able to do better. Regards, Robin Stuart

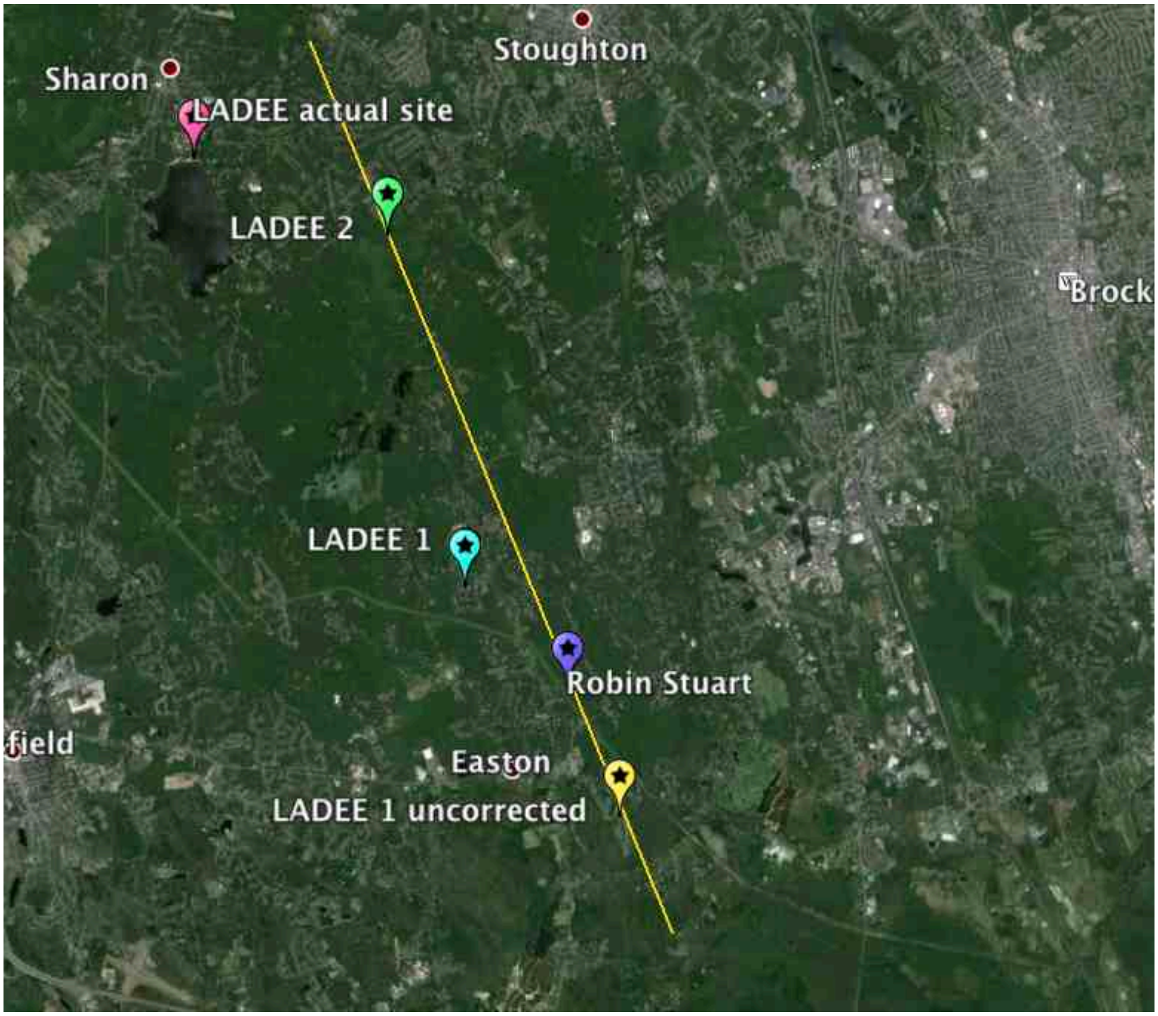


Figure 1

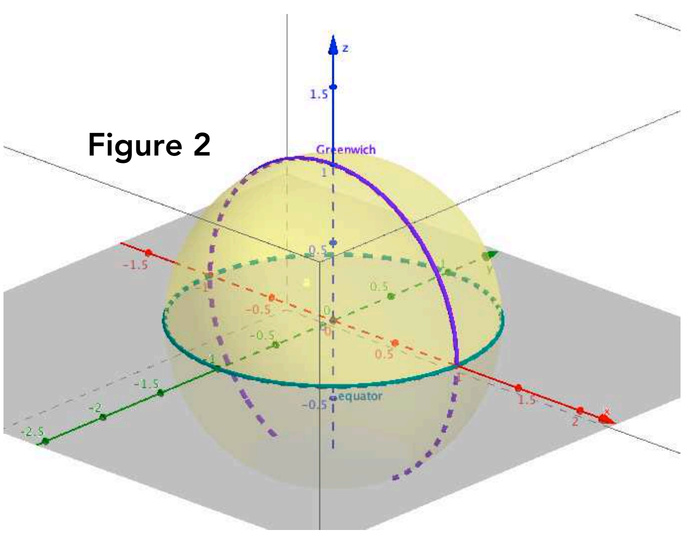


Figure 2

Background

At the given time, we have the position of the probe over the Earth, and the direction to that location. The direction is parallel to a vector from the center of the Earth, pointing to a spot on the infinitely distant celestial sphere.

As shown in Figure 2, we let the center of the Earth be the origin of a rectangular three-dimensional coordinate system. The *x*-axis intersects the equator at the Greenwich meridian, the *y*-axis intersects the equator at longitude 90º E, and the *z*-axis goes through the north pole. The Earth will be modeled as a sphere with unit radius, though we will adjust the radius to account for its oblateness when defining the position vector **p** of the probe at the time of the observation.

Once we have the *unit length* direction vector **u**, and a known point **p** on the line, the equation of the line in three-dimensional space is

$$\mathbf{x} = \mathbf{p} + t\,\mathbf{u} \qquad (1)$$

where *t* is a scalar parameter that determines the location of point **x** along the line. At the point where this line intersects the Earth's surface, its distance from the center of the Earth, i.e. the origin, will be one (ignoring the oblateness of the Earth). In other words, at the intersection point we have

$$\begin{aligned}
&|\mathbf{x}| = r_0, \quad r_0 = 1 \\
&|\mathbf{x}|^2 = r_0^2 \\
&(\mathbf{p} + t\,\mathbf{u})^{\mathrm{T}}(\mathbf{p} + t\,\mathbf{u}) = r_0^2 \\
&|\mathbf{p}|^2 + 2\mathbf{u}\cdot\mathbf{p}\,t + t^2|\mathbf{u}| = r_0^2 \\
&t^2 + (2\mathbf{u}\cdot\mathbf{p})\,t + \left(|\mathbf{p}|^2 - r_0^2\right) = 0
\end{aligned} \qquad (2)$$

We solve the quadratic in (2) for *t*, and use this value in (1) to find the location of the observer. Note that only one of the two solutions for *t* is relevant.

Translating to Rectangular Coordinates

We wish to translate the celestial coordinates of the direction vector to rectangular coordinates. The first step is convert the time of the observation to UT (universal time).

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Launch time 6 September, 2013 23:27 EDT. Stage 3 burn out was to occur
at T +207.62s
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Add four hours to convert Eastern Daylight savings Time to UT. Add an additional 3^{m} 27.62^{s} to adjust to the time of the observation. We find the time is

2013 Sept 7, 03:30:27.62 Z   Aries 38º 59.9'   (3)

Also given in (3) is the Greenwich Hour Angle (GHA) of the *first point of Aries*, from which *right ascension* is measured in an easterly direction. This GHA is taken from the Nautical Almanac or similar resource (e.g. see http://aa.usno.navy.mil/data/docs/celnavtable.php), and corresponds to longitude W 38º 59.9', or W 38.9983º.

The next step is to convert the right ascension to a corresponding longitude on Earth. At 15º per hour, 23h 05^{m} 47.4^{s} easterly from Aries corresponds to 54^{m} 12.6^{s} to the west, or 13.5525º. Adding this to Aries, the direction (as an angular distance, longitude) from the Greenwich meridian is W 52.55083º. The given declination -31º 46.1' (angular distance above or below the equator) directly translates to latitude S 31.7683º.

Now convert the spherical coordinates (*radius r, longitude θ, latitude φ*) to rectangular coordinates of the form <x,y,z> using the formula

$$(r,\theta,\varphi) \Leftrightarrow \langle x,y,z\rangle = \langle r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\varphi\rangle \quad (4)$$

(For background, see http://tutorial.math.lamar.edu/Classes/CalcIII/SphericalCoords.aspx, but recognize that for our case latitude is measured from the equator).

For the specific coordinates (1, -52.55083º, -31.7683º) we find

u= <0.5170, -0.6750, -0.5265> (5)

NOTE: The values shown here have been rounded to four decimals. However, the actual computations were carried out using double precision, approximately 16 decimal digits.

Position Vector at Burn Out

The position vector of the spot over the Earth where burn out occurred is defined by `Latitude 37.41, Longitude -69.43, Altitude 161.99km.` Letting the radius, i.e. length, be designated as *r* for the moment, the position vector in spherical coordinates, is (*r*, -69.43, 37.41). We now need to find *r* to convert to rectangular coordinates.

Radius of the Earth

See https://en.wikipedia.org/wiki/Earth_radius for background. There we find the Earth's radius is approximately 6371 km. However, the Earth is slightly squashed, and the radius at a specific latitude ϕ is given by the formula:

$$r(\phi) = \sqrt{\frac{\left(a^2\cos\phi\right)^2 + \left(b^2\sin\phi\right)^2}{\left(a\cos\phi\right)^2 + \left(b\sin\phi\right)^2}}$$

$$a = 6{,}378.1370 \text{ km (equatorial radius)} \quad (6)$$

$$b = 6{,}356.7523 \text{ km (polar radius)}$$

Evaluating (6) at latitude ϕ=37.41°, the radius of the Earth is 6370.3 km. The probe is 161.99 km above the surface, or 6523.29 km from the center. In units of the Earth's nominal radius 6371 km, the length of the position vector is r= 1.0253.

The position vector in spherical coordinates is (1.0253, -69.43, 37.41). We use (4) to find the position vector **p** of burn out in rectangular coordinates:

$$\mathbf{p}=\langle\ 0.2861,\quad -0.7625,\quad 0.6229\rangle \tag{7}$$

The geometry is depicted in the Figure 3, (created by Geogebra, see http://www.geogebra.org).

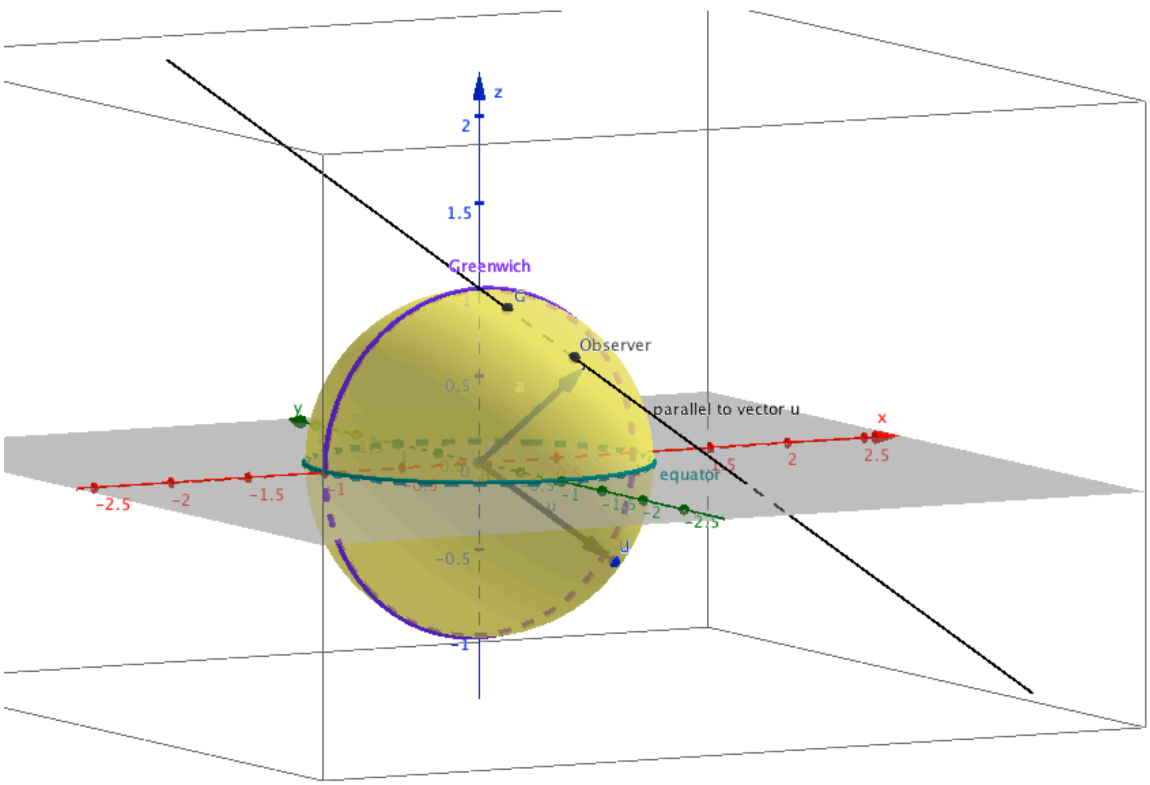

Figure 3

Solving for the Observer Position

Referring to (2), we can now solve for t in

$$t^2 + 2(\mathbf{p} \cdot \mathbf{u})\, t + \left(|\mathbf{p}|^2 - r_0^2\right) = 0\,, \quad r_0 = 1 \tag{8}$$

Computing the dot product of vectors **p** and **u** (see http://tutorial.math.lamar.edu/Classes/CalcII/DotProduct.aspx), and the magnitude squared of **p**, i.e. **p**· **p**, to find the corresponding the numerical values in (8), we get

$$t^2 + 0.6693\, t + 0.0513 = 0 \tag{9}$$

The solutions (see http://www.purplemath.com/modules/solvquad4.htm) are

t= -0.0882 and t= -0.5810 (10)

We want the first value (the second value corresponds to point G in the figure).

Using (10) in (1), we find the position vector of the observer is

x= <0.2405, -0.7029, 0.6693> (11)

To find the corresponding latitude and longitude we translate (11) to spherical coordinates, inverting the process defined by (4). The z component of **x** is $r \sin\phi$, and we find the latitude by

$$\phi = \operatorname{asin}\left(\frac{z}{|\mathbf{x}|}\right) \qquad \text{latitude} \tag{12}$$

We can use the y component $r \cos\phi \sin\theta$ (the y component will give the correct minus sign for west longitude):

$$\theta = \operatorname{asin}\left(\frac{y}{|\mathbf{x}|\cos\phi}\right) \qquad \text{longitude} \tag{13}$$

In this case, because we solved for the condition that |**x**|=1, we can ignore the divide by the magnitude of **x**. But in general, we first find

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \tag{14}$$

In this first iteration, we find the observer's position is N 42.0167º, W 71.1100º, depicted in Figure 1 as LADEE 1 uncorrected.

Now that we have an estimate of the observer's latitude, we can calculate the radius of the Earth, and use this value in equation (8) to find a revised location.

Revised Observer Location

Evaluating (6) at latitude ϕ=42º, the radius of the Earth is 6368.6 km, and normalized by the nominal radius 6371 km, r_0= 0.9996. We use this value* in (8)

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* In the original posted solution I neglected to square the r_0 value, which introduces a small error in the result.

and recalculate the parameter t. The equation with the revised r_0 value is

$$t^2 + 0.6693t + 0.0520 = 0 \tag{15}$$

The solution of (15) is

$$t = -0.0898,\ t = -0.5795 \tag{16}$$

Again choosing the first value, the revised observer position is

$$\mathbf{x} = \langle 0.2397,\ -0.7019,\ 0.6702 \rangle \tag{17}$$

Using (12) and (13), we find the new observer position to be N 42.0983, W 71.1420, labeled LADEE 2 in Figure 1. The new position* is within 1.7 nm of the actual position, and falls nicely on a Line of Position along with the uncorrected position, and Robin Stuart's position.

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* The second mistake I made was neglecting to divide by |**x**| when translating to spherical coordinates. The effect of these mistakes was to put the observer location at LADEE 1 in Figure 1.