MORE ON THE MPP
(The Maximum Probability Distribution)

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[If you're allergic to math you can skip down to the top of page 2 to read about the MPP plots.]

The NavList has seen a lot talk about cocked-hat fixes and probability distributions. Someone on the List (now I can't find who that was) mentioned that one edition of Bowditch talked about the MPP between a LOP and a fix, each having a different standard deviation (SD) from different types of measurements. An example would be a LOP with a SD of say 3nm, and a DR position with a larger SD due to (possibly) a log run.

I also found this discussion in my 1962 ed. of Bowditch (p. 686), Chapter 29, called Navigation Errors. And found it very easy to see why, and what, he's talking about. The concept and the calculation is simple in this case, and should give insight to more complicated cases, like the observations of several LOPs. Consider a point (a fix or DR) with a given SD, call it σ_0, and a single LOP with a different SD, call it σ. The distance, we'll call D, is along a line drawn from the point along the perpendicular to the LOP. The circular symmetry of the distribution around the point, and the linear distribution along the LOP tells us that the MPP lies on this line of length D. The distance along this line from the LOP we'll call x. Then the distance of this given point from the fix is (D – x). Because the two probabilities are independent of each other, the probability that a location satisfies both of them together is the product of the two normal distributions:

P = exp[-(D-x)^2 / ($2\sigma_0$^2)] * exp[-(x^2 / 2σ^2)] Eq. 1

OK, this may look complicated in this typeset, but we can see that this probability P is max when the sum of its exponents are minimum. That is

When (D-x)^2 / (σ_0^2) + (x^2 / σ^2) is minimum in x. Or (from a tiny bit of 1st semester calculus) we can set

(D-x)/σ_0^2 – x/σ^2 = 0.

Solving this for x/D, the fractional distance of the MPP from the LOP, gives

x/D = σ ^2 / (σ^2 +σ_0^2). Eq. 2

Now every humble geek checks results in as many ways possible. Here we note that:

1. Dimensions. Every quantity must have the correct dimensions. We can't measure the length of a boat in dimensions of time (and we don't measure probability in the units of length!). Both sides of our above equation are ratios of the same type quantity – both sides are dimensionless. So x/D has no dimension, x has the dimensions of D, a distance. All's OK on dimensions.

2. Next we look at special simple cases: if the LOP measurements were always perfect, their SD, σ, would be zero. And we see when $\sigma = 0$ the above equation yields x = 0. The MPP would be on the LOP. Makes perfect sense.

3. On the other hand, if instead, the SD of the fix were zero, we'd expect the MPP would be at the fix. And for $\sigma_0 = 0$ the equation gives x = D, the MPP is at the fix. So all three checks work out OK.

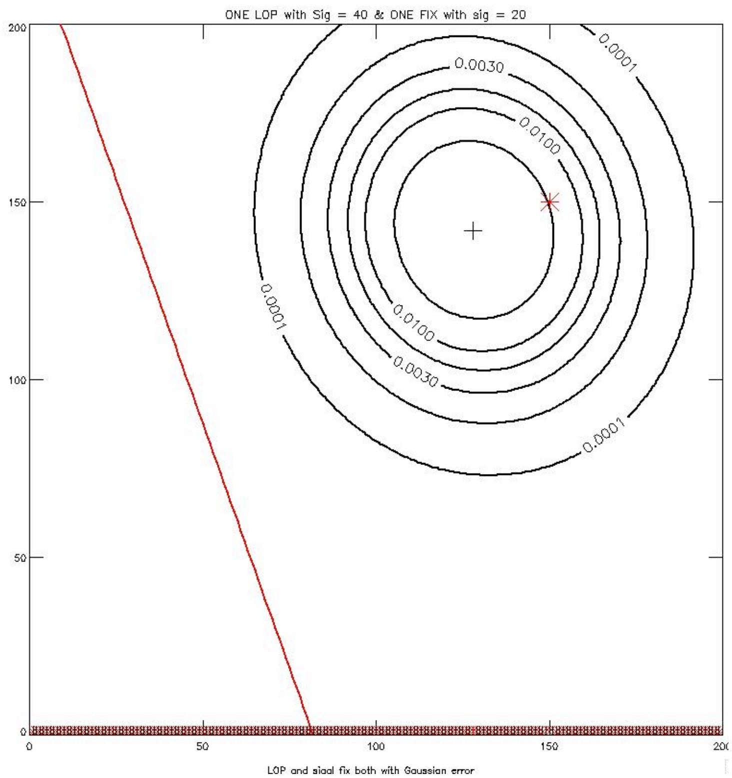
4. Also, if the two SDs are equal ($\sigma = \sigma_0$), then our Eq 2 gives x = D/2, which again makes complexly good sense.

Now for the Bowditch example, his $\sigma = 2$ and $\sigma_0 = 3$, our result gives x = 9/13 which agrees with his.
So now I (we) really understand what Bowditch is saying.

IF YOU'RE SKIPPING MATH, START HERE >>>>

Next, some plots of LOP-fix distributions and their associated the MPPs:

The positions of the LOP and fix are constant while we vary the SD of their distribution. In all plots the LOP is the red line, the fix is the red star, and MPP is the black cross. The contours are lines of equal probability per unit area. Numerically integrating over the whole area of 200x200 = 40,000 points, gives a probability of one. (In theory the area would be infinite, but I have an old cheap iMac.) The SDs are measured in the same units as the Lat-Lon axis.



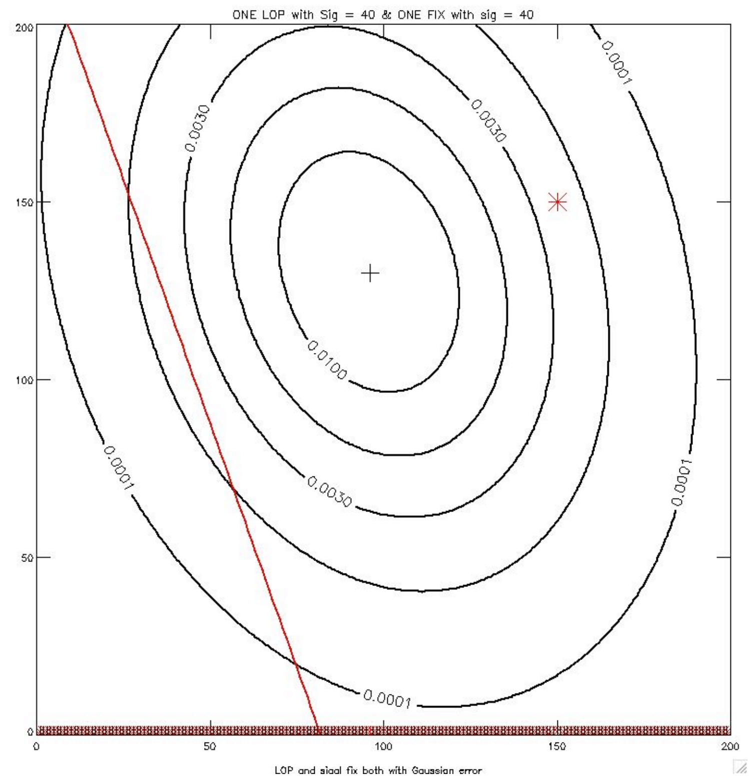


Fig 1. Here, the LOP has a SD of 40; the fix has half that SD of 20. The innermost contour here is 0.02 probability per unit area (not marked on the plot).

Our Eq. 2 gives x = 4/5 of D from the LOP for the location of the MPP. This checks by measuring from the plot. Note that the circular P distribution of the fix is just barely distorted from the linear distribution of the LOP. Numerical integration inside the inner most contour (P = 0.02) gives a 55% probability that the true location is inside that contour. This leaves a 45% probability that the true position is outside that inner contour.

Fig 2. Now the SD of both the LOP and the fix distributions are 40. With equal SDs the MPP is placed right in the middle between the LOP and the fix. We see the circular symmetry of the fix distribution is significantly distorted by the linear distribution of the LOP. The MPP is half way between the LOP and the fix, just as we saw our Eq, 2 above gave. The probability of our ship being inside the inner-most contour is now only 33% leaving an 67% for being outside.

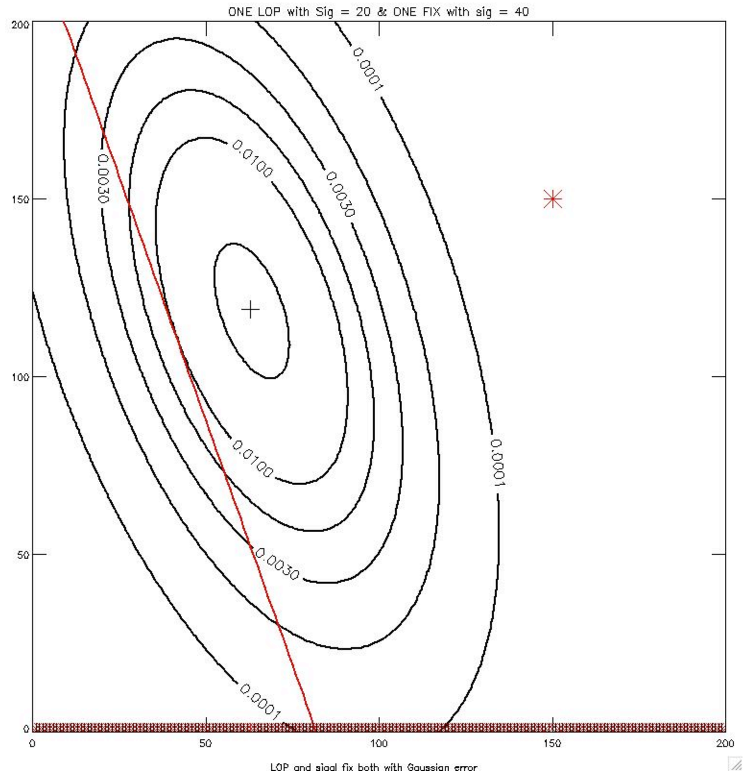


Fig 3. Finally we've reversed the two SDs, with the LOP having the more reliable SD = 20, and the fix having the SD of 40. The linear distribution of the LOP now has the major influence on the contour's shape. And our Eq. 2 now gives the MPP location at 1/5 of D, which you can see checks out with the plot.

The probability that the ship is inside the inner-most contour (the 0.02 level) is 12%. So it much more probable that the true location is outside at 88%. It's important to realize that this greater probability for the ship being outside is because the outside area is so much greater. The probability of the ship being outside the 0.01 contour is 43% and 57% inside.

The same is true for that darned old cocked hat. The smaller the hat is, the less chance the true location is inside. This is obviously true in any navigation problem – the smaller the area (of any shape) the less chance the true location is inside it. As the area becomes vanishingly small, the probability per unit area remains finite. So the product of probability-times-area must decrease to zero.

We can also see from the discussion leading to Eq. 2 that any arrangement and number of LOPs and fixes, all with normal distributions, locate their MPP by minimizing the sum of the squares of the distances from them to the MPP.

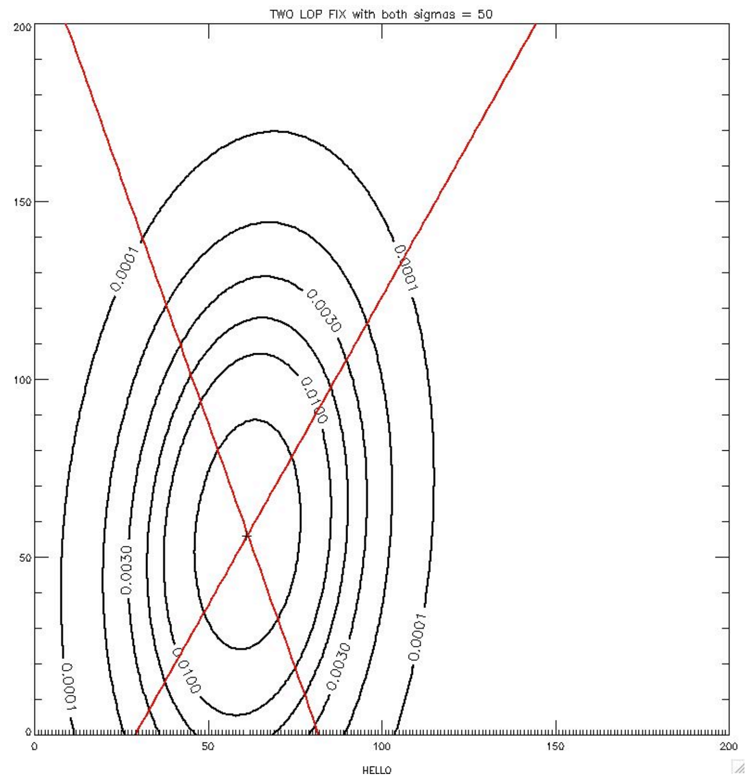


Fig. 4. Now the fix is replaced with a new LOP having a SD = 20, the same as the first LOP to the left.
Of course the MPP is at the LOP intersections. And even if the SDs were different, the MPP would still be at their intersection. But the contours would be shaped elliptically with the longer axis tipped toward the LOP having the wider distribution. In this plot the probability that the ship is inside the inner contour (at 0.02) is 89%.

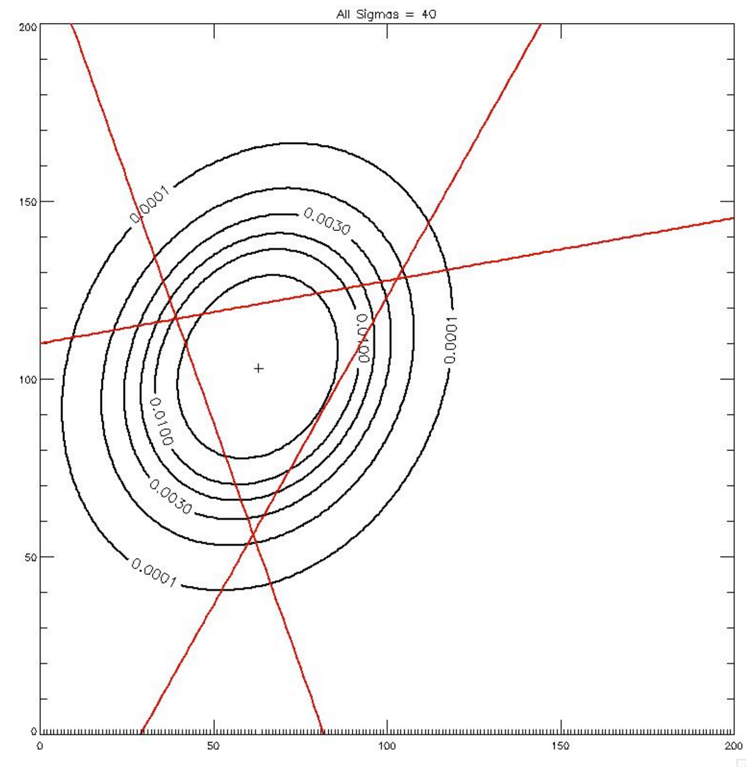


Fig. 5. Welcome to the cocked hat. Here the added third LOP has the same SD =20 as the other two. The MPP has moved to the hat's symmedian, which uses the same criteria as we've been discussing – the spot that minimizes the sum of the squared distances to the LOPs. Now the probability that the ship is inside the 0.02 contour (with area about the same as the hat) is 98%.

Ready for some Bayesian gook?? The Bayesian approach is a logical approach to inference: use all available information, assume nothing, and make no contradictions. In our discussion and plots we used the fact that previously we have made repeated altitude observations (from known locations with the same instrument & observer) that resulted in normal distributions of a determined SD. Then we're forced to center these known normal distributions on the observed LOPs. Yes, there's a good chance that these LOPs aren't really located at each distribution's maximum. But in the lack of other information we have no better inference. So the MPP we've used is the best we can do with the knowledge we have. Sounds like the Bayesian spirit.

But when we add the third LOP, forming a cocked hat, we're forced into a contradiction: all three LOPs can't be centered on their individual distributions, each having zero error. If true, they'd all intersect at a common point. On the other hand, there is no justification to choose any one of the three intersections over the other two. (Equivalently, no justification to choose one LOP over the others, as Byron Franklin seems to.) But if we choose the symmedian for the common location of all three LOPs, we're assuming, with no further information at are disposal, that each of the three altitude measurements were the best we could do. Only when we add a fourth LOP do we start to collect some new statistics. That's when the Bayesian idea comes in, but with only the most minimal addition of data, one LOP, would that Bayesian result differ from our least squares??

BTW, as I've pointed out before, the Bayesian logic is missing in the traditional running fix where the previous LOP is advanced according the estimated ship's speed and course, completely honoring DR information in one irrelevant direction while ignoring it in another direction. Hardly logical inference.