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THE HAVERSINE IN NAUTICAL ASTRONOMY.

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Amongst the more important developments in the accepted methods of nautical astronomy that have taken place in recent years may be mentioned the increased use of formulæ which involve the particular function of the angle known as the "haversine." To the mathematician pure and simple the term is almost unknown, for he is apt to attach no great amount of importance to the practical solution of triangles by means of tables, but in nautical calculations the tabulation of values of the haversine shortens and simplifies the work very materially in the solution of both plane and spherical triangles, and amongst an increasing circle of mariners the haversine methods enjoy a well-deserved popularity. It is the object of this paper to attempt to show how recent improvements in the form of these tables may possibly lead to still further simplification.

First, then, to answer a question with which a certain distinguished "wrangler" from Cambridge astonished a nautical pupil, "What is a haversine?"

The term "haversine" is merely an abbreviation of half versine, the versine being the defect of the cosine from unity.

Thus

$$\text{hav}\, A = \tfrac{1}{2}(1 - \cos A) = \sin^2 \frac{A}{2}.$$

Or the haversine of an angle is the square of the sine of half the angle, so that the Haversine Table is sometimes called the "Sine Square" Table.

To appreciate the advantage which such a table affords we have only to write down the formula for a plane triangle,

$$\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc},$$

or the analogous formula for a spherical triangle,

$$\sin^2 \frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}.$$

Evidently if a Table of Haversines is employed we shall be saved in the first instance the trouble of dividing the sum of the logarithms by two, and in the second place of multiplying the angle taken from the tables by the same number. This is the special advantage of the form of table first introduced by Professor Inman, of the Portsmouth Royal Naval College, nearly a century ago. So far as nautical astronomy is concerned the use of the Haversine Table has to do chiefly with two cases of spherical-triangle solution:

(a) Given three sides of a triangle to find an angle, the case to which reference is made above.

(b) Given two sides and the angle included, to find the third side.

Under one or other of these heads nearly all the operations which occur in the ordinary daily routine at sea may be said to be comprised.

The second problem has been brought into more than ordinary prominence of late years by the growing favour of the "Marcq" position-lines, upon the principle of calculated altitudes suggested by Admiral Marcq-Saint-Hilaire of the French Navy some thirty-five years ago. It is this method which the ingenious tables of Lieutenant Radler de Aquino, described in a recent number of the PROCEEDINGS, are intended to facilitate.

The principles upon which the Saint-Hilaire process is based were so clearly set forth in the paper above referred to, that we may perhaps confine ourselves here to the solution of the necessary triangle.

Let us suppose that in lat. 43° 20′ N, the declination of the sun being 18° 36′ N, and the hour angle $3^h\ 46^m$ W, it is required to find the zenith distance.

The work will be as follows:

L sine polar distance (71° 24′)	9.97670
L sine colatitude (46° 40′)	9.86176
L hav hour angle ($3^h\ 46^m$)	9.35031
(Sum) L hav θ	9.18877
θ	46° 17′
Natural versine θ	.308907
Natural versine (pol. dist. − colat.) 24° 44′	.091735
(Sum) versine zen. dist.	.400642
Zen. dist.	53° 10.5′

This is the working of the problem upon the model that has been followed in the Royal Navy for the last 80 or 90 years. It will be observed that the process consists of two parts; first, the determination by means of logarithmic sines and haversines of an auxiliary angle θ. The value of θ so found is carried as an argument to the Table of Natural Versines, and its versine, added to that of the difference between polar distance and colatitude, gives the versine of the zenith distance required.

Thus three tables in all are required, involving logarithmic sines and haversines, and natural versines. In the year 1905, however, Mr. P. L. Davis, of the British Nautical Almanac Office, published a volume of tables,* in which was included a new and improved Table of Haversines, the use of which would enable the Table of Natural Versines to be eliminated, the number of tables required being thus reduced to two.

The leading feature of this table is that it gives side by side, in parallel columns, the L haversine and Nat haversine, the latter being printed in heavier type than the former. Thus, in the case of the auxiliary angle 46° 17′, as determined above, we find:

Angle.	L Hav.	Nat Hav.
46° 17′	9.18880	.15445

Making use of this improved table we should finish the work thus:

L hav θ 9.18877	Nat hav	.15445
	Nat hav (P. D. – colat.)	.04587
	(Sum) Nat hav Z. D.	.20032
		Z. D. = 53° 10.5′

Here we have no concern with the numerical value of θ, which has no longer to be carried as an argument to the Table of Natural Versines, and an unnecessary opening of the tables is thereby avoided. The Table of Natural Versines, moreover, becomes redundant, and so obvious are the advantages of the Davis Table that in the latest edition of Inman's Tables, the official text-book used in the Royal Navy, the Table of Natural Versines is omitted altogether, and the combined Table of Logarithmic and Natural Haversines is adopted in its place.

* Requisite Tables. By Percy L. Davis, F. R. A. S. London, J. D. Potter. Price, 10s. 6d.

It has occurred to the writer that the process of amendment might be carried still farther, and that not only the problem worked above, but all the ordinary processes required in every-day nautical astronomy might be dealt with by means of the single improved Haversine Table alone. Such a work as the "Requisite Tables" of Mr. Davis might then be limited to this Haversine Table, which occupies 105 pages, and the remaining 72 pages, which include the logarithms of numbers and of sines, cosines, and tangents, might be dispensed with, in their present form, their place being taken by the Tables of Meridional Parts, Corrections in Altitude, Increase of R. A. of Mean Sun, etc., perhaps some 25 pages in all. The tables omitted would be such as are required only for occasional use, and having no special nautical character, would be found, when wanted, in any general collection of logarithmic and mathematical tables. The advantage of so limiting the size of the volume of tables in daily use will be obvious to all computers.

What then is proposed is briefly this: To obtain a general formula for the spherical triangle, involving only the haversine, and to show how this formula may be adapted to the solution of the following general types:

TYPE I.—Given two sides and the included angle, to find the third side.

TYPE II.—Given three sides to find an angle.

And further, to show how the expressions which result may be applied to the solution of the ordinary practical nautical problems:

(a) Given colatitude, polar distance, hour angle, to find the zenith distance.

(b) Given polar distance, zenith distance observed on a small bearing, and hour angle, to find meridian zenith distance.

(c) Given colatitude, polar distance, and zenith distance, to find hour angle.

(d) Given colatitude, zenith distance, polar distance, to find the azimuth, or true bearing.

(e) Given colatitude and polar distance, to find the amplitude.

To Establish the Necessary General Formula.

In the spherical triangle ABC we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Therefore

$$1-\cos A=\frac{\sin b \sin c-\cos a+\cos b \cos c}{\sin b \sin c}.$$

Therefore

$$\sin b \sin c \operatorname{vers} A=\cos (b-c)-\cos a.$$

or

$$\sin b \sin c \operatorname{vers} A=\operatorname{vers} a-\operatorname{vers}(b-c).$$

Dividing each side of the equation by two,

$$\begin{aligned}\operatorname{hav} a-\operatorname{hav}(b-c) &=\sin b \sin c \operatorname{hav} A\\ &=\tfrac{1}{2}\{\cos(b-c)-\cos(b+c)\}\operatorname{hav} A\\ &=\tfrac{1}{2}\{\operatorname{vers}(b+c)-\operatorname{vers}(b-c)\}\operatorname{hav} A\\ &=\{\operatorname{hav}(b+c)-\operatorname{hav}(b-c)\}\operatorname{hav} A,\end{aligned}$$

or

$$\operatorname{hav} a=\operatorname{hav}(b-c)+\{\operatorname{hav}(b+c)-\operatorname{hav}(b-c)\}\operatorname{hav} A \qquad (1)$$

Here, then, we have an expression containing haversines only, and involving the sides a, b, c with one angle, in this case the angle A.

We will proceed to apply it in the solution of the Types I and II. Since no other function than the haversine is involved, the constant repetition of the word may be avoided, it being understood that:

By $L(\theta)$ is meant L hav θ; i. e., the tabular log haversine of θ.
By $N(\theta)$ is meant N hav θ, or the natural haversine of θ.

TYPE I.—*Given the Sides b, c and the Angle A to Find a.*

In the general formula (1) let

$$\operatorname{hav} \theta=\operatorname{hav}(b+c)-\operatorname{hav}(b-c). \qquad (2)$$

Then, substituting in equation (1),

$$\operatorname{hav} a=\operatorname{hav}(b-c)+\operatorname{hav} \theta \operatorname{hav} A. \qquad (3)$$

To find θ we have, from (2),

$$N(\theta)=N(b+c)-N(b-c). \qquad (4)$$

This determines θ.

Again, let

$$\operatorname{hav} \phi=\operatorname{hav} \theta \operatorname{hav} A$$

in equation (3). Then

$$L(\phi)=L(\theta)+L(A). \qquad (5)$$

This determines ϕ.

Thus equation (3) becomes

$$\text{hav}(a) = \text{hav}(b-c) + \text{hav}(\phi),$$

where b, c, ϕ are known, and

$$N(a) = N(b-c) + N(\phi). \qquad (6)$$

TYPE II.—*Given Three Sides, to Find an Angle A.*

From the equation (1),

$$\{\text{hav}(b+c) - \text{hav}(b-c)\}\,\text{hav}\,A = \text{hav}\,a - \text{hav}(b-c). \qquad (7)$$

Let

$$\text{hav}\,a - \text{hav}(b-c) = \text{hav}\,\theta, \qquad (8)$$

$$\text{hav}(b+c) - \text{hav}(b-c) = \text{hav}\,\phi. \qquad (9)$$

Then, from equation (8),

$$N(\theta) = N(a) - N(b-c)\,; \qquad (10)$$

and from (9),

$$N(\phi) = N(b+c) - N(b-c). \qquad (11)$$

These equations determine θ and ϕ, and from equation (7)

$$L(A) = L(\theta) - L(\phi). \qquad (12)$$

Application to the Problem: Given Colatitude, Polar Distance and Hour Angle, to Find Zenith Distance.

This is the general problem of the Marcq position-lines, and presents a straightforward example of Type I. The polar distance (p) corresponds to b, colatitude to c, hour angle (h) to A, and zenith distance (z) to a.

The formulæ will be:

$$N(\theta) = N(p+c) - N(p-c),$$
$$L(\phi) = L(\theta) + L(h),$$
$$N(z) = N(p-c) + N(\phi).$$

Application to the Problem of Reduction to Meridian, Approximate Colatitude being Supposed Known.

The formulæ and general procedure will be very much the same as in the preceding problem, except that in the third equation we must transpose z and $p-c$. For one problem is practically the converse of the other. In the first case we know p and c, and have to find z. In the second, z being known we determine an accurate value of $(p-c)$, which at first is only approximately known. The third equation, therefore, is written in this case

$$N(p-c)=N(z)-N(\phi),$$

the angle ϕ being calculated as before.

Application to the Problem of Finding Hour Angle, Colatitude, Polar Distance, and Zenith Distance being Known.

From the formulæ of Type II,

$$N(\theta)=N(z)-N(p-c),$$
$$N(\phi)=N(p+c)-N(p-c),$$
$$L(h)=L(\theta)-L(\phi).$$

Application to the Altitude Azimuth; Colatitude, Polar Distance and Zenith Distance being Known, to Find the True Bearing (Z).

This again illustrates Type II. For b we have z; for a, the polar distance p is substituted; c represents the colatitude; and for A we have Z.

Then

$$N(\theta)=N(p)-N(z-c),$$
$$N(\phi)=N(z+c)-N(z-c),$$
$$L(Z)=L(\theta)-L(\phi).$$

It is unnecessary, perhaps, to consider in detail the time azimuth, because it is evident that from p, c and h, the data of the problem, we may find z under the formulæ of Type I, and then from p, c and z determine Z by Type II.

Application to the Amplitude.

The interest in this problem, which indeed may be regarded as only a special case of the altitude azimuth, is mainly theoretical, since by means of the Azimuth Tables we may find the compass error with equal accuracy and less trouble at almost any time of day. It is, however, easy to obtain a simple special haversine formula as follows: Let A represent the amplitude.

Since

$$\sin^2 A=\frac{\sin^2 d}{\sin^2 c},$$

we have

$$L(2A)=L(2d)-L(2c).$$

PRACTICAL EXAMPLES.

The following practical examples from Raper's Practice of Navigation (Nineteenth Edition) will serve to illustrate the various processes.

Given Colatitude, Polar Distance, and Hour Angle, to Find Zenith Distance.

Ex. (Raper, p. 238).

$c=67° 45'$, $p=92° 49'$, $h=2^h\ 14^m\ 36^s$. Required z.

$N(p+c)$	160° 34′	.97151
$N(p-c)$	25 4	.04709
(Diff.) $N(\theta)$		.92442
$L(\theta)$		9.96586
$L(h)$	$2^h\ 14^m\ 36^s$	8.92315
(Sum) $L(\phi)$		8.88901
$N(\phi)$		.07745
$N(p-c)$	25° 4′	.04709
(Sum) $N(z)$		.12454
z		41° 19.7′

Given Polar Distance, Hour Angle, Zenith Distance Near Meridian, and Approximate Colatitude, to Find Reduction to Meridian.

Ex. (Raper, p. 251).

c (approx.) $=84° 57'$, $p=73° 45'$, $z=11° 49'$, $h=15^m\ 40^s$.

$N(p+c)$	158° 42′	.96585
$N(p-c)$	11 12	.00952
(Diff.) $N(\theta)$		.95633
$L(\theta)$		9.98060
$L(h)$	$15^m\ 40^s$	7.06736
(Sum) $L(\phi)$		7.04796
$N(\phi)$		.00111
$N(z)$	11° 49′	.01060
(Diff.) $N(p-c)$		.00949
$(p-c)$		11° 11′

Thus the meridian zenith distance is 11° 11′, the reduction being 38′, which is the value obtained by Raper, who makes use of a "second reduction" involving the use of sine and tangent tables, as well as a constant logarithm.

Given Colatitude, Polar Distance, and Zenith Distance, to Find the Hour Angle.

Ex. (Raper, p. 220).

Given $c=38°\ 50'$, $p=70°\ 33'$, $z=52°\ 9'$, to find h.

	$N(z)$ 52° 9′	.19320		$N(p+c)$ 109° 23′	.66594
	$N(p-c)$ 31 43	.07467		$N(p-c)$ 31 43	.07467
(Diff.) $N(\theta)$		.11853	(Diff.) $N(\phi)$		.59127
$L(\theta)$		9.07381			
$L(\phi)$		9.77178			
(Diff.) $L(h)$		9.30203			
h		$3^h\ 32^m\ 47^s$			

Given Colatitude, Polar Distance and Zenith Distance, to Find Azimuth.

Ex. (Raper, p. 241).

Given $c=38°\ 30'$, $p=69°\ 58'$, $z=49°\ 35'$ W, to find Z.

	$N(p)$ 69° 58′	.32872		$N(z+c)$ 88° 5′	.48328
	$N(z-c)$ 11 5	.00933		$N(z-c)$ 11 5	.00933
(Diff.) $N(\theta)$		.31939	(Diff.) $N(\phi)$		.47395
$L(\theta)$		9.50432			
$L(\phi)$		9.67572			
(Diff.) $L(Z)$		9.82860			
Z		N 110° 21′ W			

Given Colatitude and Polar Distance to Find the Amplitude (A).

Ex. (Raper, p. 327).

$c=73°$, $d=23°$ N. Sun setting.

$L(2d)$ 46°	9.18376
$L(2c)$ 146	9.96119
(Diff.) $L(2A)$	9.22257
$2A$	48° 14′
A	W 24° 7′ N

General Observations upon the Haversine Methods Proposed.

The practice of nautical astronomy, as carried out in the daily routine on board ship at sea, may be broadly defined as the discovery, by means of the observed altitude of some heavenly body, of the coördinates latitude and longitude, which determine the position of a ship, and of the error of the compass by which her course is directed. To these may be added, in the case of disciples of the Saint-Hilaire School, the calculation of an altitude for an assumed latitude and longitude. These problems are solved by different writers in a great variety of ways, involving for each its own particular set of formulæ. The expression made use of for finding hour angle from three sides, for instance, is in general a perfectly different formula from that employed in calculating zenith distance, and the two require entirely independent demonstrations. Under the system outlined above, which I would venture to style the "All-Haversine" System of Nautical Astronomy, we have but to obtain one single general formula, which can be easily adapted to the conditions of the particular case by any one familiar with the elementary properties of logarithms. The necessary course of theoretical study required of nautical students, who are not always overburdened with mathematical knowledge, is somewhat simplified thereby.

In such a subject as this, however, the importance of the theoretical portion is somewhat overshadowed by that of the practical side, so it may be well to consider the matter under this aspect also. Leaving out of consideration the amplitude, which is the subject of special treatment, the four great problems of every-day navigation divide themselves into two groups, according as they come under Type I, which includes the Reduction to Meridian and calculation of Zenith Distance; or under Type II, which involves the calculation of Hour Angle for Longitude, and True Azimuth for Compass Error.

With regard to Type I, the formulæ bring out the fact, which has perhaps not received sufficient attention in the past, that the two problems are essentially similar—and indeed almost one and the same problem. The three equations employed are indeed identical in the two cases, the only point of difference being that if we require zenith distance, as in determining a Marcq position-line we write the third equation

$$N(z) = N(p - c) + N(\phi),$$

while if the meridian zenith distance is required the equation will be set down as

$$N(p-c)=N(z)-N(\phi).$$

In the two cases we obtain our first auxiliary angle θ by a common process from the formula

$$N(\theta)=N(p+c)-N(p-c).$$

This angle is converted into ϕ by the introduction of h in the formula

$$L(\phi)=L(\theta)+L(h),$$

and we have then only to consider whether z or $(p-c)$ is the quantity required, and handle the third equation accordingly.

The two problems, therefore, amount to little more than one, some little discrimination being required at the end of the work, according to the data of the problem.

With regard to the formulæ which come under Type II, where, three sides being given, we have to determine an angle, it may be claimed for the method here suggested that it is at least as short, as simple, as easy to acquire in the first instance, and, when acquired, to retain in the memory, as any process to be found in text-books ancient or modern.

In the case of the hour angle, we have but to subtract $N(p-c)$ first from $N(z)$, and then from $N(p+c)$, to obtain the two auxiliary angles θ and ϕ.

Then from $L(\theta)-L(\phi)$ we have at once $L(h)$, the same openings of the tables which give $N(\theta)$, $N(\phi)$ supplying $L(\theta)$ and $L(\phi)$ without any necessity to take into account the actual values of the auxiliary angles. The same observation applies to the calculation of azimuth, the only point of difference being that z and p are transposed in the latter problem.

An essential feature of the work under either type is the absence of what is known in certain public examinations as "long tots." The operations involved are of the simplest character, never requiring more than the addition or subtraction of two lines of figures. Every computer is aware of the tendency of mistakes to creep in, when the summation of four or five lines has to be effected, as in most of the ordinary methods.

Lastly, with special reference to the particular function of the angle employed, some advantage results from the use of the haversine, in preference to the more ordinary ratios such as sine, cosine, etc., from the circumstance that the haversine increases continuously from 0° to 180°, and is always positive. It follows from this that the Haversine Table is easier to handle than those which deal with sines and cosines, and we are saved from any risk of taking out L sin 54° from the right-hand column, in lieu of L sin 36°, which should be found in the left-hand column of the page, as is often done by careless computers.

Generally it may be said of the "all-haversine" system here proposed that it demands but one formula, one function of the angle, one Table of Combined Haversines, natural and logarithmic. It was a particularly happy idea of Mr. Davis to combine them in this manner, and a considerable simplification in the calculation of Marcq position-lines has already resulted from the combination. Some day, perhaps, we shall all be enthusiastic followers of the Saint-Hilaire system, but meanwhile there remain a certain percentage who cling to their "Longitude by Chronometer" and "Latitude by Reduction" problems, and it is partly in their interest that the writer has been led to ask permission to set forth these notions, in the hope of further extending the sphere of usefulness of this excellent table.