PHASE CORRECTION FOR VENUS

W. Robert Bernecky
September, 2018

I Introduction

The positions of the stars and planets in the sky (the ephemeris) typically refer to the geometric center of the object, which is accurate for visually point-like stars. But for planets, specifically Venus, with an apparent diameter of up to sixty-six arc seconds[1], the "center" of the lighted region may be measurably distant from the geometric center. Figure 1 shows examples of how the phase of the planet, as observed from Earth, determines how much of its surface is illuminated.

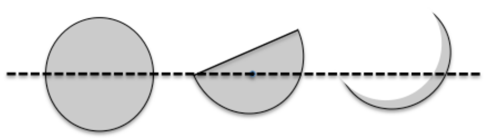

Figure 1 - gray is illuminated

An observer, using a sextant, measures the altitude of the body above the sea horizon, placing the center of the body on the horizon. When the observer has little or no magnification, the center of a defocused blur is placed on the horizon, which may introduce perhaps a few arc minutes of error as compared to using the geometric center. To compensate for this error, we can provide the center of the illuminated region as the position of the planet — this is the so-called phase corrected position.

II Center of an Illuminated Body

Definition of Center

The first problem is to define what we mean by the center of the lit region of a planet and how it may vary with the magnification available to the observer. I have not researched the problem, but have adopted as "reasonable" to treat the lit region (as seen from Earth) as a two-dimensional flat figure, and will place the visual center at the centroid, i.e. at the center of gravity of the figure.

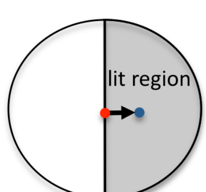

Figure 2

It is self-evident that the geometric and visual centers coincide when the body is full, and as some of the body becomes shaded, the visual center moves toward the illuminated area, along the center-line of the body, as indicated in figure 2. In the extreme, the visual center moves to the edge of a crescent.

Illuminated Fraction of a Disk

Define a baseline from the planet to the Sun. Now define the <u>phase angle *i*</u> as the smallest angle off this baseline to point toward Earth. Note that we restrict *i* to the interval 0° to 180° inclusive. When the phase angle *i*=0°, an observer on Earth sees a fully illuminated disk. (Though, for Venus and Earth, this would mean the Sun sits between them and Venus would not be visible at all, being "behind" the Sun.) At *i*=180°, the disk presents its dark side to the Earth, and is not visible.

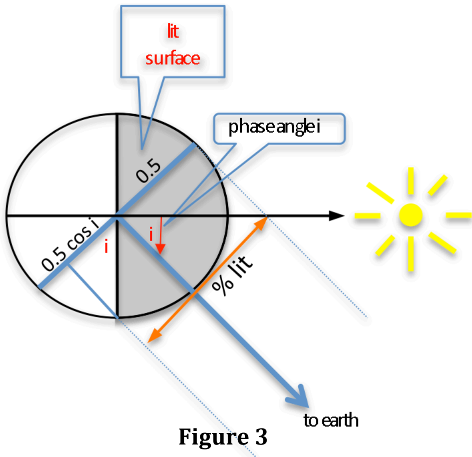

Figure 3

In figure 3, we set the diameter of the disk to one, and its radius to 0.5, and we see that the lit length of the diameter is

$$0.5 + 0.5\cos i \qquad (1)$$

ranging from one down to zero as *i* ranges from 0° to 180°.

Treating *k* as a ratio of lit length to total length, we have the formula[2]:

$$k = \frac{1+\cos i}{2} \qquad (2)$$

It turns out that *k* is both the fraction of the diameter that is illuminated, and also the ratio of lit area to the total area. See the appendix for details. Also note that as *i* goes beyond 90°, the quantity 0.5 cos *i* becomes negative, and *k* becomes smaller than 0.5, indicating a waning planet.

Centroid of the Illuminated Disk

Figure 4 shows the three-dimensional picture. Figure 5 presents the two-dimensional view as seen from the Earth. The lower red arc in figure 5 demarks the illuminated upper region from the dark region below. The red arc is one-half of an ellipse (see the appendix for details), with a semi-major axis of *r* (i.e. the radius, or if you prefer, the semidiameter of the planet) and a semi-minor axis *b*.

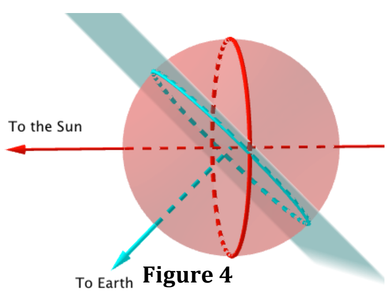

Figure 4

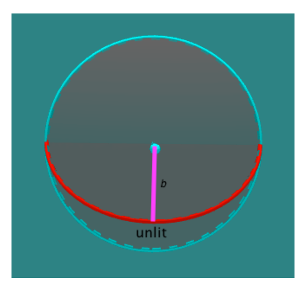

Figure 5

Given a circle of radius *r* and center at (0,0), the area and centroid[3] of its upper half (a semicircle) is

$$\bar{x} = 0, \quad \bar{y} = \frac{4r}{3\pi}, \quad Area = \frac{\pi r^2}{2} \qquad (3)$$

The area and centroid for the lower semiellipse, with semi-major axis *r* and semi-minor axis *b* is

$$\bar{x} = 0, \quad \bar{y} = -\frac{4b}{3\pi}, \quad Area = \frac{\pi r b}{2} \qquad (4)$$

The combined "center of gravity" for the composite body is found using[4]

$$\bar{y} = \frac{\bar{y}_1 \cdot Area_1 + \bar{y}_2 \cdot Area_2}{Area_1 + Area_2} \qquad (5)$$

which gives us

$$\bar{x} = 0$$

$$\bar{y} = \frac{\left(\frac{4r}{3\pi} \cdot \frac{\pi r^2}{2} - \frac{4b}{3\pi} \cdot \frac{\pi r b}{2}\right)}{\left(\frac{\pi r^2}{2} + \frac{\pi r b}{2}\right)} = \frac{\frac{2r}{3}\left(r^2 - b^2\right)}{\frac{\pi r}{2}\left(r + b\right)} = \frac{4\left(r - b\right)}{3\pi} \qquad (6)$$

In (6), the quantity *r-b* is the length of the unlit part of the diameter. From (2), we know the length of the lit diameter is

$$k \cdot diameter = k2r$$

and consequently, the length of the unlit diameter is

$$r - b = 2r - 2rk = (1 - k)2r \qquad (7)$$

Using (7) in (6), we have the distance to move from the center of the disk <u>in the direction of the lighted region</u>:

$$\bar{y} = \frac{8r}{3\pi}(1 - k) \qquad (8)$$

III Finding the Phase Correction

Fraction of Venus that is Illuminated

It will take us too far afield to get into a discussion on how to find the positions of the planets, but software is available that provides position vectors of the three bodies in question, namely Venus, the Sun, and Earth. Assume we have these position vectors, labeled **p**, **q** and **r** in figure 6.

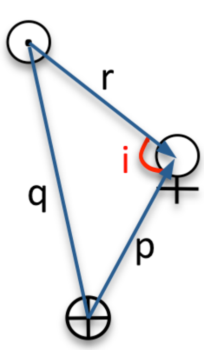

Figure 6

The phase angle *i* (see figure 3) can be found using the Law of Cosines. Let the lower case letters represent the lengths of each vector, then by the Law of Cosines:

$$q^2 = p^2 + r^2 - 2pr\cos i \qquad (9)$$

from which we get

$$\cos i = \frac{p^2 + r^2 - q^2}{2pr} \tag{10}$$

Also, from (2), we can write

$$\cos i = 2k - 1 \tag{11}$$

and using (11) to eliminate the reference to the cosine in (10)

$$2k - 1 = \frac{p^2 + r^2 - q^2}{2pr}$$

$$k = \frac{1}{2}\left(\frac{p^2 + r^2 - q^2}{2pr} + 1\right) = \frac{p^2 + 2pr + r^2 - q^2}{4pr} \tag{12}$$

$$k = \frac{(p + r)^2 - q^2}{4pr}$$

where p is the Earth-Venus distance, q is the Sun-Earth distance, r is the Sun-Venus distance, and k is the illumination fraction (from 0 to 1) of Venus, as seen from Earth[2].

Venus Semidiameter

Meeus[5] states that the 8.34" semidiameter for Venus provided by the Astronomical Almanac refers to the planet's crust, and not to the top of the reflecting cloud layers. He suggests using the older value of 8.41". PyEphem[6] (free software) uses 8.46". In either case, this value is the semidiameter s_0 of the planet in arc seconds when observed from a distance of 1 AU (149597870700 m). At any other distance p measured in AU, the observed semidiameter[5] is

$$s = \frac{s_0}{p} \cdot \frac{\pi}{180 \cdot 3600} \quad \text{radians} \tag{13}$$

where we have translated from arc seconds to radians.

Phase Correction Formula

Assume we have a <u>unit vector **p**</u> at the center of the Earth pointing at the center of the "flat disk" of Venus, as observed from Earth. We could rotate the vector by some angle θ so that it points at the desired new location. Perhaps easier, we can <u>add a properly scaled vector **c**</u> (correction vector) that is both perpendicular to **p** and points to the "lit region" of the flat disk. Lastly, re-normalize the resultant to unit length (though for the small angles involved here, this may be unnecessary).

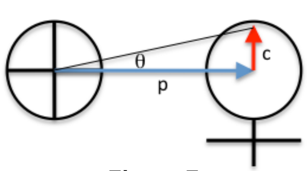

Figure 7

From figure 7, with the vectors having lengths *p* and *c*, we can write

$$\tan\theta = \frac{c}{p} \tag{14}$$

and for θ<<1 radian, and the length of **p** equal to one, we have the approximation

$$\theta[rads] \approx c \tag{15}$$

In other words, we should scale the <u>unit vector **c**</u> by a correction *c* measured in radians. The expression for the correction value is given by (8), in the same units as *r* (i.e. the semidiameter) of Venus. Thus we use the semidiameter of Venus expressed in radians (eq 13) in place of *r*:

$$\begin{aligned} \mathbf{p}' &= \mathbf{p} + \frac{8s}{3\pi}(1-k)\,\mathbf{c} \\ \mathbf{p}_{\text{lit}} &= \frac{\mathbf{p}'}{|\mathbf{p}'|} \end{aligned} \tag{16}$$

where *k* is the illuminated fraction, defined in (12).

Correction Vector c

We now find the unit length vector **c** that points in the direction to apply the phase the correction. To begin, we note that the Sun, Earth and Venus define a plane, and that plane is defined by a vector normal to it. This normal can be computed by a vector cross-product. Referencing the vectors, <u>scaled to unit-length</u>, depicted in figure 6, we have

$$\mathbf{n} = \mathbf{r} \times \mathbf{p} \tag{17}$$

where **n** will be vector pointing out of the paper, toward the reader. If we now cross **n** with **p**, the resultant vector **c** will be perpendicular to **p**, and lie in the plane of the three bodies, and pointing toward the lighted region of Venus, viewed as a flat disk. Normalize **c** to unit-length before using it in (16).

The operations just described are known as a triple vector product, and it has a convenient identity:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a} \tag{18}$$

which replaces three (relatively complicated) cross products with simple dot products and vector addition. In our problem, we have

$$\begin{aligned} \mathbf{c}' &= (\mathbf{r} \bullet \mathbf{p})\mathbf{p} - \mathbf{r} \\ \mathbf{c} &= \frac{\mathbf{c}'}{|\mathbf{c}'|} \end{aligned} \tag{19}$$

knowing that **p** is unit length.

If **p** and **r** are parallel then their cross product has zero length, so to avoid numerical problems the algorithm should abort and return a zero phase correction. As this condition only occurs when Venus is "behind" the Sun or when completely dark, this is a reasonable thing to do.

Components of the Phase Correction

The new look direction $\mathbf{p}_{\text{lit}}$ given by (16) points at the centroid of the illuminated area of Venus. Assuming $\mathbf{p}_{\text{lit}}$ has components

$$\mathbf{p}_{\text{lit}} = \left[\cos dec \cos ra \quad \cos dec \sin ra \quad \sin dec\right] \tag{20}$$

we translate to spherical coordinates using

$$\begin{aligned} dec &= \sin^{-1} \mathbf{p}_{\text{lit}}[3] \\ ra &= \operatorname{atan2}\left(\mathbf{p}_{\text{lit}}[2], \mathbf{p}_{\text{lit}}[1]\right) \end{aligned} \tag{20}$$

If we also translate the original **p** vector to spherical coordinates, we can explicitly compute the phase corrections to the declination and right ascension:

$$\begin{aligned} \Delta_{ra} &= ra_{lit} - ra_{orig} \\ \Delta_{dec} &= dec_{lit} - dec_{orig} \end{aligned} \tag{21}$$

IV An Example

Using PyEphem[6] (which I don't recommend due to bugs, but nevertheless is convenient for this example), at the date and time of 1977 March 18 00:02:17 UT, we find

Description	Distance (AU)	Unit Vector
Earth→Venus **p**	0.3326292634010315	[0.90257558, 0.32622803, 0.28094946]
Sun→Earth **q**	0.9953900575637817	
Sun→Venus **r**	0.7192550897598267	[-0.96492143, 0.2111272 , 0.15605105]
Correction Dir **c**		[0.43033829, -0.70314989, -0.56602932]

$$k = \frac{(p+r)^2 - q^2}{4pr} = 0.1208585$$

$$s[rads] = \frac{s_0["]}{p[AU]} \cdot \frac{\pi}{180 \cdot 3600} = \frac{8.41}{0.3326292634} \cdot \frac{\pi}{180 \cdot 3600} = 0.0001225774$$

$$\mathbf{p}' = \mathbf{p} + \frac{8s}{3\pi}(1-k)\,\mathbf{c}$$

$$\mathbf{p}' = \mathbf{p} + 9.14719729213\text{e-}05\ \mathbf{c} = [\ 0.90261494,\ 0.32616371,\ 0.28089768]$$

	Original (°)	Phase Adj. (°)	Correction (')
Right Ascension	19.87194512375141	19.86753517695265	-0.26460
Declination	16.31687933886024	16.31378825824658	-0.18546

Translating from right ascension to Greenwich Hour Angle negates that correction:

$$GHA = Sidereal\_Time_{Greenwich}° - (RA+\Delta)° = GHA_0 - \Delta$$

We find the phase corrections to be:

$$\Delta GHA = +0.265' \approx +0.3'$$

$$\Delta Dec = -0.185' \approx -0.2'$$

On 1977 March 18 00:02:17 UT, sidereal time at Greenwich is 11:44:03.26, or 176.0136°, and the GHA of Venus, using the phase adjusted location, is

$$GHA = 176.0136° - 19.867535° = 156°\ 8.764' \approx 156°\ 8.8'$$

and the corrected declination is

$$Dec = 16.313788° = 16°\ 18.827' \approx 16°\ 18.8'$$

Appendix

Here I will fill in some details glossed over in the main section, as often what is obvious when working on a problem is less so at some later time, and I am left wondering, "How did I get that?"

Projection of a Tilted Circle is an Ellipse

When looking at the lighted orb of Venus, but at some tilt angle, we see an apparently flat disk, some fraction of which is unlit. In figure A-1, the vertical black diameter represents a plane (its other dimension is in/out of the paper) tilted *i* degrees relative to the "blue" plane (which is the "screen" that the observer is viewing).

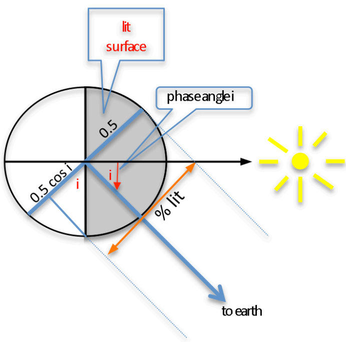

Figure A- 1

Figure A-2 is a three-dimensional version.

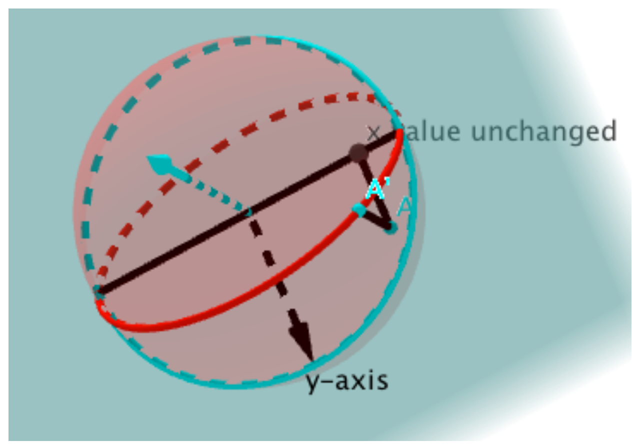

Figure A- 2

The red circle in the plane tilted by angle *i* relative to the blue plane has the equation

$$x'^2 + y'^2 = r^2 \qquad \text{(A1)}$$

A point A' on this circle is projected onto the "blue screen" at point A, with the same x-coordinate as A', but with a y-coordinate foreshortened:

$$\begin{gathered} x = x' \\ y = y' \cos i \Rightarrow y' = \frac{y}{\cos i} \end{gathered} \qquad \text{(A2)}$$

Using these values in (A1), we have

$$\begin{gathered} x^2 + \frac{y^2}{\cos^2 i} = r^2 \\ \frac{x^2}{r^2} + \frac{y^2}{r^2 \cos^2 i} = 1 \end{gathered} \qquad \text{(A3)}$$

which can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(A4)}$$

which is the equation of an ellipse with

$$\begin{aligned}&\text{semi}-\text{major axis } a = r\\&\text{semi}-\text{minor axis } b = r\cos i\end{aligned} \qquad \text{(A5)}$$

Also, we note without proof that the area of the full ellipse is

$$A = \pi r b \qquad \text{(A6)}$$

Illumination Fraction k

In the text, the ratio of the lit portion of the diameter to the entire diameter was introduced:

$$k = \frac{r \pm b}{2r} = \frac{1 \pm \frac{b}{r}}{2} \qquad \text{(A7)}$$

where r is the radius of the disk, and b is the distance from the center to the limit of the lighted region. As noted, b depends on the phase angle i:

$$b = r\cos i \qquad \text{(A8)}$$

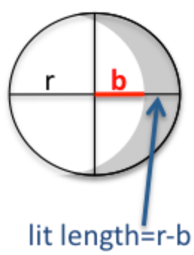

Figure A- 3

Using (A8) in (A7), we have the result presented in the text

$$k = \frac{1 \pm \frac{r\cos i}{r}}{2} = \frac{1 + \cos i}{2} \qquad \text{(A9)}$$

Note that the cosine automatically gives the correct sign, and we need not explicitly denote the plus or minus on b. Below, we can assume that b has the correct sign, as determined by the phase angle i. For example, in figure 3, where the phase angle i is about 120°, b is negative, and "eats into" the illuminated area.

The area of the illuminated region is the area of a semicircle, plus or minus a half-ellipse with semi-major axis r and semi-minor axis b:

$$A_{lit} = \frac{\pi r^2}{2} + \frac{\pi r b}{2} \qquad \text{(A10)}$$

and the ratio of A_{lit} to the area of the whole circle is

$$\frac{A_{lit}}{A} = \frac{1}{\pi r^2}\left(\frac{\pi r^2}{2} + \frac{\pi r b}{2}\right) = \frac{1 + \frac{b}{r}}{2} = k \qquad \text{(A11)}$$

Finding the Centroid of a Semiellipse

The centroids of common two-dimensional figures are listed at
https://en.wikipedia.org/wiki/List_of_centroids

However, it may be of interest to show how to calculate it from first principles.

The centroid is the "average" *x* and *y* values, and for a uniform density figure, it is the center of mass. Each coordinate is computed independently, using the formula

$$\bar{y} = \frac{1}{A} \iint_{Region} y \, dA \tag{A12}$$

where A is the total area of the figure.

In our case, the figure is the upper half of the ellipse shown in figure A-4. We place its semi-major axis to coincide with the *x*-axis, and its semi-minor axis with the *y*-axis. Its area is one-half that of a full ellipse:

$$A = \frac{\pi r b}{2} \Rightarrow \frac{1}{A} = \frac{2}{\pi r b} \tag{A13}$$

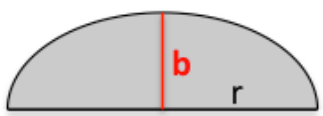

Figure A- 4

We now find its centroid using (A12). In (A12), the double integral ranges over *y*, whose limits must be a function of *x*, and over *x*, -*r* to +*r*. The lower limit of *y* is zero. The upper limit depends on the *x* value, which we can find using the equation of an ellipse:

$$\frac{x^2}{r^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \frac{x^2}{r^2}} \tag{A14}$$

Plugging (A13) and (A14) into (A12)

$$\bar{y} = \frac{2}{\pi r b} \int_{x=-r}^{r} \int_{y=0}^{b\sqrt{1-\frac{x^2}{r^2}}} y \, dy \, dx \tag{A15}$$

Doing the inner integral

$$\int_{y=0}^{b\sqrt{1-\frac{x^2}{r^2}}} y \, dy = \left.\frac{y^2}{2}\right]_0^{b\sqrt{1-\frac{x^2}{r^2}}} = \frac{b^2}{2}\left(1 - \frac{x^2}{r^2}\right) \tag{A16}$$

and the outer integral

$$\bar{y} = \frac{2}{\pi r b} \int_{x=-r}^{r} \frac{b^2}{2}\left(1 - \frac{x^2}{r^2}\right) dx = \frac{b}{\pi r}\left.\left(x - \frac{x^3}{3r^2}\right)\right]_{-r}^{r} = \frac{4b}{3\pi} \tag{A17}$$

This puts the *y* value of the centroid about 0.424 of *b* above the *x*-axis. By symmetry (or intuition), we know the *x* value of the centroid is zero, but it can be calculated using (A12) with *y* replaced with *x*.

References

[1] Venus Observational Parameters found at https://nssdc.gsfc.nasa.gov/planetary/factsheet/venusfact.html

[2] J. Meeus, Astronomical Algorithms, 2nd ed. Willman-Bill, Richmond VA 1998, pg 283.

[3] https://en.wikipedia.org/wiki/List_of_centroids

[4] https://en.wikipedia.org/wiki/Centroid#By_geometric_decomposition

[5] J. Meeus, ibid, pgs 389-390.

[6] https://en.wikipedia.org/wiki/PyEphem