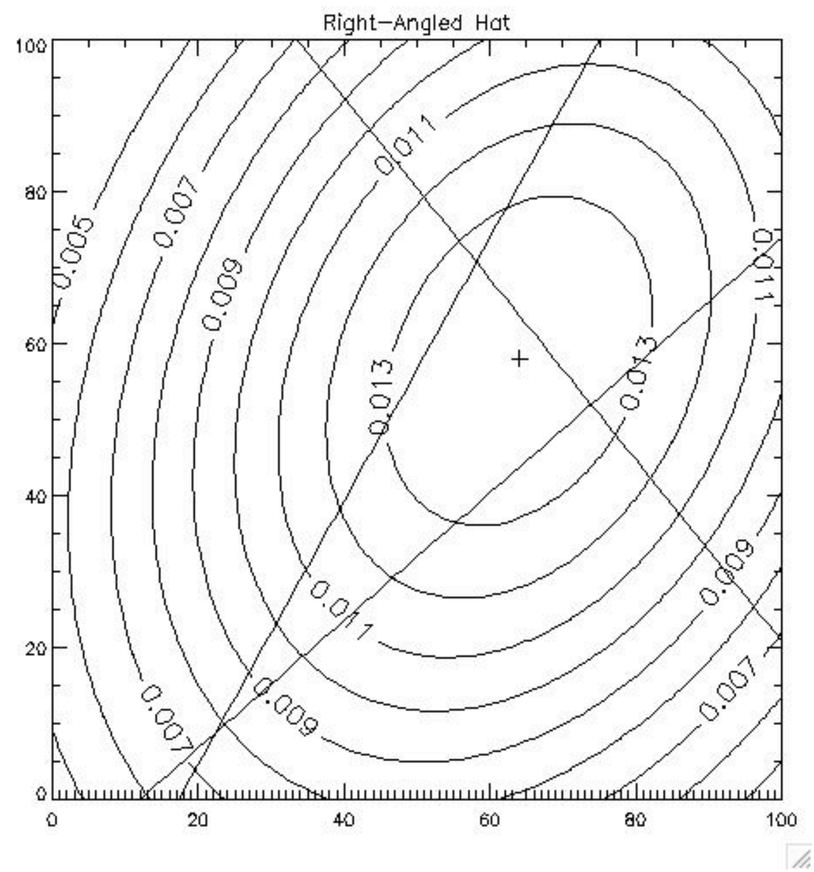
That Darned Old Cocked Hat
(Expanded from NavList Post 14687, John Karl)

In any estimation problem we use our best knowledge, make no assumptions, and allow no contradictions. One common case is the 3-LOP fix problem when we know a priori that all LOPs have a normally distributed (i.e., Gaussian, or bell-shaped) random error, all with the same known standard deviation, σ. The LOP's azimuths are calculated exactly. Now we ask three questions about the cocked hat that is formed by these three LOPs: (1) where is the point having the maximum probably per area, called the Maximum Probability Point, the MPP, (2) what is that probability, and (3) what is the probability that the ship is located outside of the cocked hat?

The probably per area is $P(x,y) = (1/A) \exp(-(d1^2 + d2^2 + d3^2)/2 \sigma^2)$

where d1, d2, and d3 are the perpendicular distances to the three LOPs, and A is a normalizing factor (the integral of P(x,y) over the entire 2D surface – over all infinity). So we see that the point of maximum probability minimizes the sum squared distances from the three LOPs. And as I later learned from Herbert Prinz, this is called the symmedian point.

The figure below shows the contours of P(x,y) of a 100x100 grid with a σ of 60 units. The x-y scales are arbitrary and relate only to σ. For example, if we can consider that the grid is 10x10 nm, then the standard deviation in the LOPs is 6 nm. The cross marks the MPP.



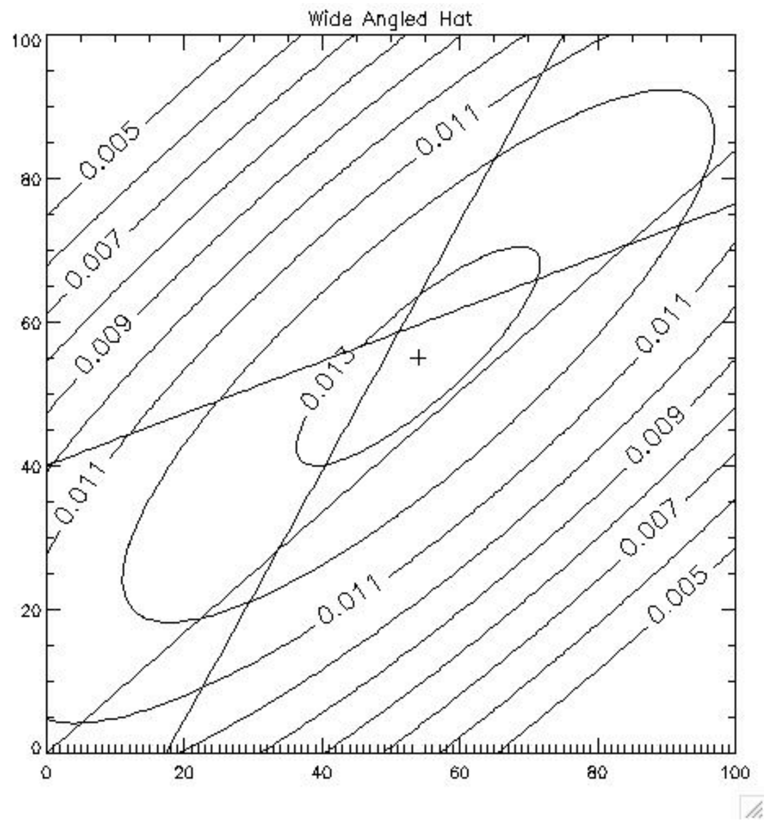
We can easily see some interesting results:

1. First, the probability at the MPP is 0.014/area (not shown), a quite small value because the total area involved is so large (theoretically, infinite).

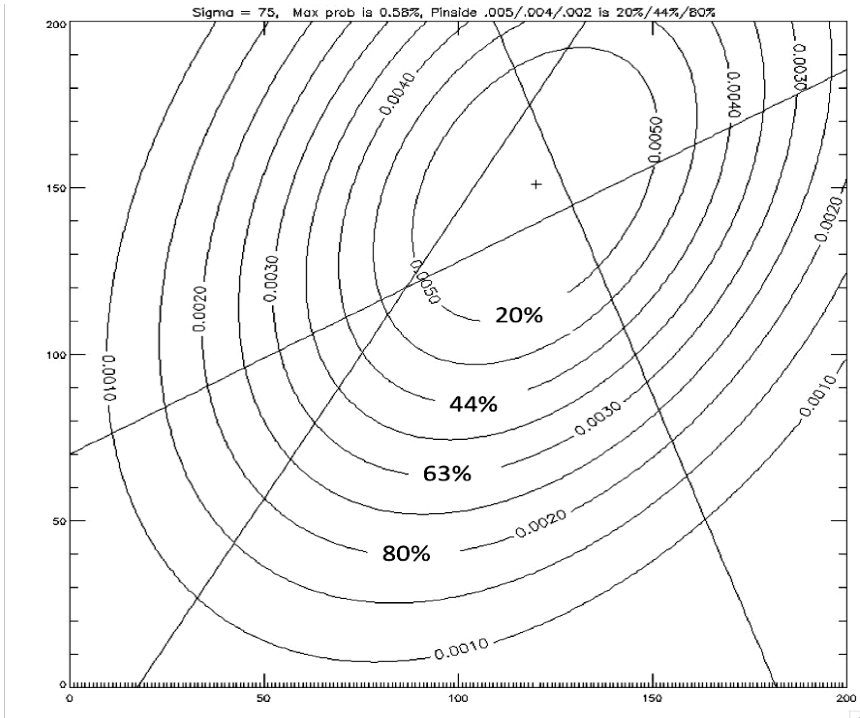
2. Second, the probability that the ship is outside of the cocked hat is about 84%. Think about what this means for trying to use the max probability location in practical navigation.

3. Third, the location of the MPP is none of the commonly mention ones: It's not the center of gravity, it's not the Fermat point (which minimizes the sum of the distances to the hat's vertices), it's not the point that bisects the hat's three interior angles, and it's not the Steiner point. It's the symmedian point, as we observed above.

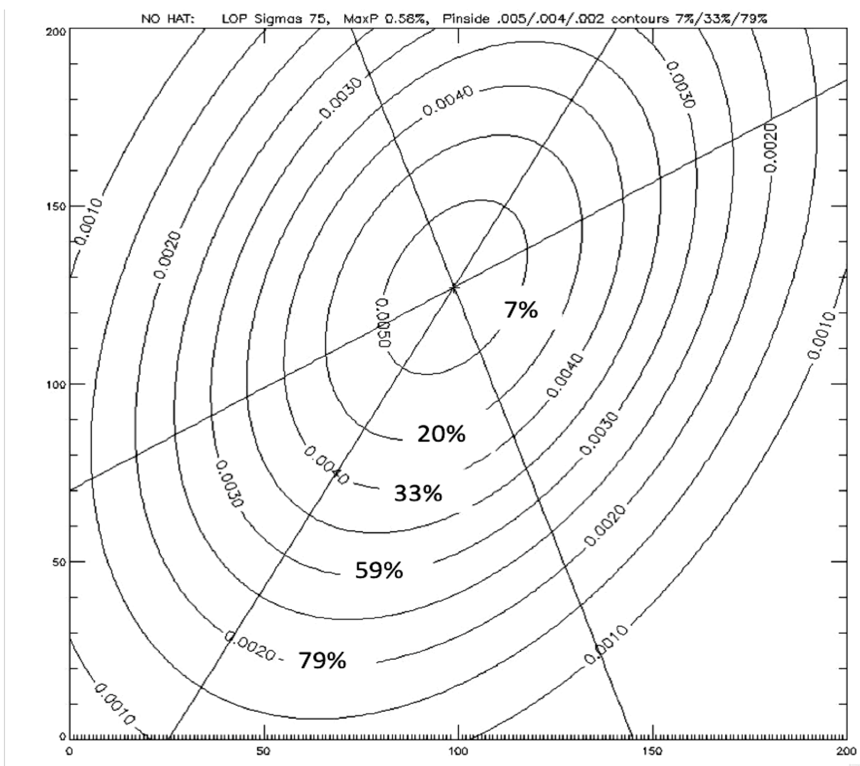
Next, the figure below of a wide-angled hat shows that even when one of the interior angles is greater than 120°, the location of the MPP is still inside the hat, unlike what we've read in some places. (But still the probability that the ship is outside the hat is about 94% in this example.) From thinking of the P(x,y) equation above, we can easily see that the MPP, the one that minimizes the sum squared distance to the LOPs, will always be located inside the cocked hat.



Obvious from these figures is that as the hat gets smaller, even though the max probability increases, the probability of being inside the hat decreases – it even decreases to zero as the hat shrinks to a point because the area inside the hat goes to zero. So the smaller the hat, the smaller the chance that the ship's inside it!



In the two figures above and below, the bold percentages on the contours are the probability that the fix is inside those contours. Above we see a relatively good fix, with the hat being well within the individual LOP's standard deviation of 75 units in this case.



Here all three LOPs cross at the same point, with no hat. Note that the 20% probability contour is about the same size in both plots above. Perfect LOP crossings don't seem to help much – surprised???