

We shall proceed to illustrate these formulæ by some practical examples.

*Example 1.* By equal altitudes the errors of a chronometer, on local mean time at noon, were found to be as follows:—

	Chron. fast	<sup>h</sup> <sup>m</sup> <sup>s</sup>	Differences.
May 3d		3 17 4·55	
5th		3 17 14·00	$a = + 9·45$
8th		3 17 24·90	$b = + 20·35$
9th		3 17 29·25	$c = + 24·70$
12th		3 17 41·40	$d = + 36·85$
14th		3 17 49·50	$e = + 44·95$
			$\text{Sum} = + 136·30$

Here,

	Squares.	Cubes.
$n_1 = 2$	4	8
$n_2 = 5$	25	125
$n_3 = 6$	36	216
$n_4 = 9$	81	729
$m = 11$	121	1331
Sum 33	267	2409

Also,

$n_1 a = + 18·90$
$n_2 b = + 101·75$
$n_3 c = + 148·20$
$n_4 d = + 331·65$
$m d = + 494·45$
$+ 1094·95$

Whence we have

$$A = 33$$

$$B = \frac{267}{22} = 12·136$$

$$C = 267$$

$$D = \frac{2409}{22} = 109·5$$

$$P = + 136·30$$

$$Q = + 1094·95$$

Now, by formula (6),

$$x = \frac{DP - BQ}{AD - BC}$$

Therefore, substituting numerical values as above,

$$\begin{aligned} x &= \frac{14924·85 - 13288·27}{3613·5 - 3240·3} \\ &= \frac{1636·58}{373·2} \\ &= 4·385 \end{aligned}$$

Also,

$$\begin{aligned}
 y &= \frac{P - A s}{B} \\
 &= \frac{136.30 - 144.705}{12.136} \\
 &= \frac{-8.405}{12.136} \\
 &= -0.693
 \end{aligned}$$

These values of  $x$  and  $y$  thus obtained are dependent on all the observations made during the period between the 3d and 14th, and from them the state of the rate at any moment during that period can be determined.

Thus, on May 3d, at the moment when the first observation was made,

$$\text{The rate } x = +4^s.385$$

And on May 14th, at the moment when the last observation was taken,

$$\begin{aligned}
 \text{The rate } x + y &= +4^s.385 + (-0.693) \\
 &= 3^s.692
 \end{aligned}$$

And at any other time, May  $(3+n)^{\text{days}}$ , by formula (3),

$$\begin{aligned}
 \text{The rate} &= x + \frac{n}{m} \cdot y \\
 &= +4^s.385 - \frac{n}{m} \cdot 0.693
 \end{aligned}$$

In order to determine the error of the chronometer, since we assume that equal confidence can be placed on all the observations (or otherwise they would have been rejected), we have, by taking the arithmetic mean between all the errors, the mean error corresponding to the mean of the times as a mean epoch.

Hence we find in this case,

$$\text{May } 8^{\text{d}}.5, \text{ Chron. fast } 3^{\text{h}} 17^{\text{m}} 27^{\text{s}}.27$$

To find the rate corresponding to this epoch, since May 8<sup>d</sup>.5 is 5<sup>d</sup>.5 in advance of May 3d,  $n = 5.5$ , and by our formula (3),

$$\begin{aligned}
 \text{Rate} &= x + \frac{n}{m} \cdot y \\
 &= x + \frac{5.5}{11} \cdot y \\
 &= 4^s.385 - \frac{1}{2} (0.693) \\
 &= 4^s.039
 \end{aligned}$$

This *is* interesting. Least-squares, in linear algebra, is formulated as

$$\mathbf{M}\mathbf{x} = \mathbf{b}$$

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b}$$

$$\mathbf{x} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{b}$$

where  $\mathbf{M}\mathbf{x} = \mathbf{b}$  represents  $n$  linear equations. We can redo Shadwell's examples, and see that he has muddled the procedure. My best guess as to how he arrived at his approach, is he adopted a least-squares implementation for fitting a line to a set of points:

$$y_i = mx_i + b, \text{ for } i= 1 \text{ to } n$$

In matrix-vector form this is:

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \cdot & \cdot \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_n \end{bmatrix}$$

Notice that the second column of the matrix is all ones. If you work through Shadwell's approach, this is exactly what he is assuming. However, his matrix definitely does not have all ones in its second column. Consequently his "least-squares" estimates are in error. On the other hand, with the examples he gives, the error is quite small, tenths of a second, for example. I have not looked into just how bad things can get using his approach, but it surely is not optimum.

See below for a Matlab program that implements a least-squares for one of his examples.

```

% test least-squares on chronometer errors
%   Date   Chron fast   Diff
%   May  3     4.55     ----
%         5     14.00     9.45
%         8     24.90    20.35
%         9     29.25    24.70
%        12     41.40    36.85
%        14     49.50    44.95

% In days from the first observation
% days  error
%  2    9.45
%  5   20.35
%  6   24.70
%  9   36.85
% 11   44.95

% chron is fast this many seconds
chron= [ 4.55 14.00 24.90 29.25 41.40 49.50];

n=[ 2 5 6 9 11 ]'; % interval between measurements (days)
m= n(end); % total number of days
M= [n n.*n./(2*m)] %
% Error in chronometer in seconds
p= chron'-4.55; p= p(2:end)

% equation to solve is M x = p
xx= inv(M'*M)*M'*p
str= sprintf('Base rate is %5.3f secs, changing %5.3f secs per %d days',...
    xx(1),xx(2),m);
disp(str)
str= sprintf('Rate on May 3rd %5.3f secs, on May 14th %5.3f\n',...
    xx(1),sum(xx)); disp(str);

disp('Estimated gain in secs/day of chronometer:')
rate= polyval([xx(2) xx(1)],n/m),

```

M =

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    2.0000    0.1818
    5.0000    1.1364
    6.0000    1.6364
    9.0000    3.6818
   11.0000    5.5000

```

p =

9.4500  
20.3500  
24.7000  
36.8500  
44.9500

xx =

4.2269  
-0.3072

Base rate is 4.227 secs, changing -0.307 secs per 11 days  
Rate on May 3rd 4.227 secs, on May 14th 3.920

Estimated gain in secs/day of chronometer:

rate =

4.1711  
4.0873  
4.0594  
3.9756  
3.9197