THE JOURNAL OF NAVIGATION (2017), **70**, 618–627.
doi:10.1017/S0373463316000886

Great Circles and Rhumb Lines on the Complex Plane

Robin G. Stuart

(*Valhalla, New York, USA*)
(E-mail: robinstuart@earthlink.net)

Mapping points on the Riemann sphere to points on the plane of complex numbers by stereographic projection has been shown to offer a number of advantages when applied to problems in navigation traditionally handled using spherical trigonometry. Here it is shown that the same approach can be used for problems involving great circles and/or rhumb lines and it results in simple, compact expressions suitable for efficient computer evaluation. Worked numerical examples are given and the values obtained are compared to standard references.

KEYWORDS

1. Algorithm. 2. Nautical Navigation. 3. Path.

Submitted: 9 August 2016. Accepted: 26 November 2016. First published online 25 January 2017.

1. INTRODUCTION. Many of the problems that arise in celestial navigation can be cast in the form of a rotation from one spherical coordinate system into another. For example standard Line of Position (LoP) navigation can be seen as a transformation of the Right Ascension and declination coordinates of a celestial body into the observer's local altitude and azimuth.

A number of equivalent formalisms are available to perform rotations in three-dimensional (3D) space. These include 3D rotation matrices, typically specified in terms of Euler parameters, quaternion methods and bilinear transformations of complex numbers whose coefficients are identified with the Cayley-Klein parameters (Arfken and Weber, 1995). For a survey of these methods see Heard (2006) and Stuart (2009a). Mapping points on the sphere to numbers on the complex plane by stereographic projection allows practical use to be made of the last of these formalisms for navigational purposes and was explored in a prior article (Stuart, 2009a). Key results applicable to LoP navigation are given in the Appendix. The same approach has been found to be applicable in other related disciplines (Stuart, 1984; Stuart, 2009b).

An alternative framework that uses stereographic projection to map problems in spherical trigonometry to plane trigonometry has been given by Donnay (1945).

In this paper navigational problems involving great circles and rhumb lines are considered as a natural by-product and extension of the complex number methods previously described.

Under stereographic projection all circles on the sphere map to circles on the complex plane but great circles form a subclass that is subject to additional constraints with the consequence that the characteristics of any great circle are completely and uniquely specified by the complex number lying at its centre.

A rhumb line or loxodrome on the sphere maps to a logarithmic spiral on the complex plane whose algebraic form is known and can be exploited in rhumb line or Mercator sailing problems. It should be noted that stereographic and Mercator projections are closely related. Applying the (complex) logarithm function to points on a complex plane mapped from the surface of a sphere by stereographic projection generates a Mercator projection with meridians represented by straight lines parallel to the real axis and parallels of latitude running parallel to the imaginary axis.

While not offering any clear advantages for manual calculations, the approach described here produces compact formulae that are suitable for efficient computer implementation. Complex numbers and operations on them are native to many scientific computer languages, FORTRAN, C++, PERL, Mathematica etc. Calculations performed in this way possess the following advantages:

- The two spherical coordinates (latitude/longitude; right ascension/declination; altitude/azimuth) are encapsulated into a single complex number and do not need to be carried separately through the calculations.
- Results from spherical trigonometry are replaced by purely algebraic formulae which tend to be compact and efficient to code and evaluate.
- Ambiguities as to the quadrant in which a particular angle lies are eliminated.
- Circles on the sphere map to circles on the complex plane.

In the present work stereographic projection is defined using the geometric convention that the complex plane is tangent to the south pole of the sphere with a point at latitude, L, and longitude, λ, mapping to

$$(L, \lambda) \rightarrow z = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) e^{i\lambda} \tag{1}$$

The inverse transformation is

$$L = 2\tan^{-1}|z| - \frac{\pi}{2}; \quad \lambda = \arg z \tag{2}$$

Corresponding results for the case where the complex plane is tangent to the north pole are obtained by changing the signs of both L and λ.

Following the notation of Nevanlinna and Paatero (1969) and Stuart (1984, 2009a, 2009b), the complex conjugate of z will be denoted by $\bar{z}$.

2. GREAT CIRCLES ON THE COMPLEX PLANE.

The general properties of great circles on the complex plane were derived and described in (Stuart, 1984) and are summarised here.

A great circle on the Riemann sphere under stereographic projection maps to a circle on the complex plane. The image, z_p, of a pole of the great circle on the sphere is related to its centre, z_c, on the complex plane by

$$z_c = \frac{2z_p}{1 - |z_p|^2} \tag{3}$$

and its radius, r, satisfies the relation

$$r^2 = 1 + |z_c|^2 \tag{4}$$

Points z lying on a great circle therefore satisfy an equation of the form

$$z\bar{z} - \bar{z}_c z - z_c \bar{z} - 1 = 0 \tag{5}$$

Conversely the images of the poles of the great circle are

$$z_p = \frac{-1 \pm r}{\bar{z}_c} \tag{6}$$

Hence as noted in the introduction a great circle is fully specified by the point z_c at its centre.

The great circle distance, D, between points represented by z_1 and z_2 is

$$D = 2\tan^{-1}\left|\frac{z_2 - z_1}{\bar{z}_1 z_2 + 1}\right| \tag{7}$$

The following sections give expressions for the centres and other characteristics of great circles on the complex plane that are applicable to problems in navigation.

Figure 1 shows the stereographic projection of the great circle running through Yokohama (YK) and San Francisco (SF) from the sphere onto the plane of complex numbers. The Earth is represented by a sphere of unit diameter with the plane tangent to it at the South Pole. Each point on its surface is mapped to the plane by drawing a line from the North Pole (N) through that point and continuing it on until it intersects the plane. One of the two poles of the great circle (p_1) is visible. The poles lie 90° from each point on the great circle. The northern vertex (v_1) or point of greatest latitude is also shown. The images under stereographic projection of the points SF, YK, p_1, v_1 are labelled z_{SF}, z_{YK}, z_{p_1}, z_{v_1} respectively. The centre of the great circle on the complex plane is z_c and does not, in general, correspond to the image of either of its poles. As is evident from the figure the centre of the great circle and the images of its poles and vertices all lie on a straight line passing through the origin on the complex plane.

2.1. *Great Circle passing through two Points z_1 and z_2.* Since z_1 and z_2 both satisfy Equation (5) eliminating $\bar{z}_c$ between the two equations gives

$$z_c = \frac{\left(|z_1|^2 - 1\right) z_2 - \left(|z_2|^2 - 1\right) z_1}{\bar{z}_1 z_2 - \bar{z}_2 z_1} \tag{8}$$

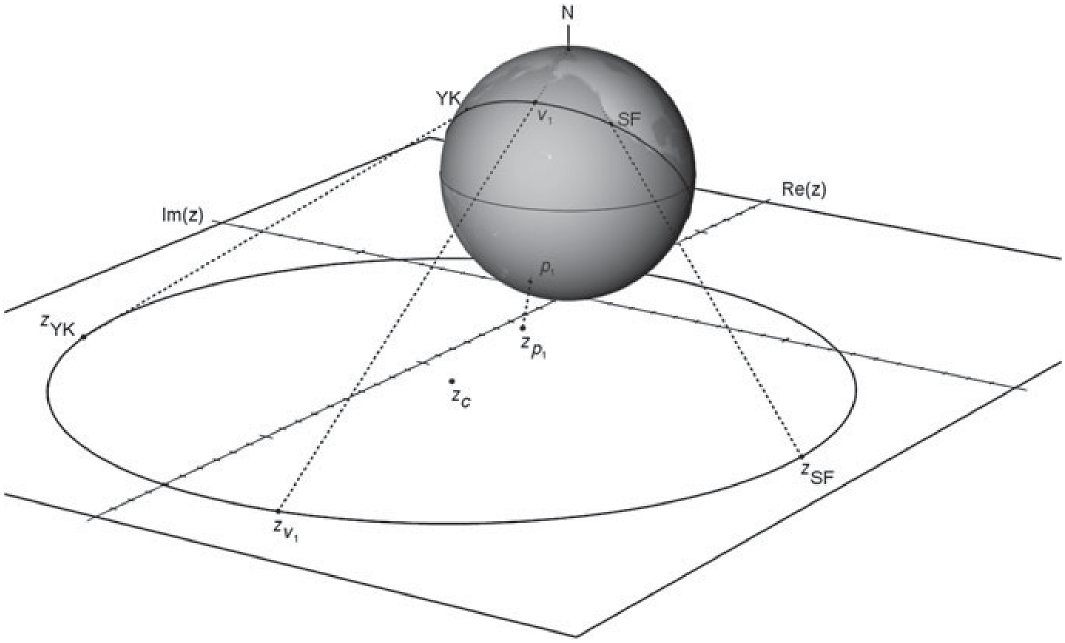

Figure 1. Stereographic projection of the great circle running through Yokohama (YK) and San Francisco (SF) from the sphere onto the plane of complex numbers.

2.2. *Great Circle Equidistant from two Points z_1 and z_2.* From Equation (7) points z lying on the great circle equidistant from z_1 and z_2 satisfy the condition that

$$\left|\frac{z - z_1}{\bar{z}_1 z + 1}\right| = \left|\frac{z - z_2}{\bar{z}_2 z + 1}\right| \tag{9}$$

This may be rearranged to yield an equation of the form of Equation (5) representing a circle with centre

$$z_c = \frac{\left(|z_1|^2 + 1\right) z_2 - \left(|z_2|^2 + 1\right) z_1}{|z_1|^2 - |z_2|^2} \tag{10}$$

2.3. *Great Circle passing through a Point z_1 at an Angle C.* The centre, z_c, of a great circle passing through the point z_1 with direction C measured eastward from north lies on a line drawn through z_1 at right angles to the direction C. It therefore takes the form

$$z_c = z_1 + kiz_1 e^{iC} \tag{11}$$

for some real constant k.

Setting $z = z_1$ in Equation (5) and using the expression above for z_c allows k to be determined yielding

$$z_c = z_1 \frac{e^{-iC} + e^{iC}\,|z_1|^{-2}}{e^{-iC} - e^{iC}} = \frac{iz_1}{2\,|z_1|}\left\{\frac{|z_1|^{-1}\,e^{iC} + |z_1|\,e^{-iC}}{\sin C}\right\} \tag{12}$$

2.4. *Point where a Great Circle crosses a Meridian.* When plotting a great circle track on a chart it can be convenient to compute points at specified values of longitude.

The point z_λ at which a great circle crosses a particular meridian of longitude, λ, can be calculated by writing $z = |z|\, e^{i\lambda}$ in Equation (5) and solving for $|z| \geq 0$. The result is

$$z_\lambda = \left\{ \mathrm{Re}\left(z_c e^{-i\lambda}\right) + \sqrt{1 + \mathrm{Re}\left(z_c e^{-i\lambda}\right)^2} \right\} e^{i\lambda} \tag{13}$$

2.5. *Points where a Great Circle crosses a Parallel.* Similarly the point z_L where a great circle crosses parallel of latitude, L, can be found using Equation (5) with the constraint that $|z_L| = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right)$ which gives

$$z_L = z_1 \left\{ 1 \pm i\sqrt{|z_L|^2 \,/\, |z_1|^2 - 1} \right\} \tag{14}$$

in which $z_1 = (|z_L|^2 - 1)/(2\bar{z}_c)$. Valid solutions exist provided $r - |z_c| \leq |z_L| \leq r + |z_c|$ where $r = \sqrt{1 + |z_c|^2}$.

2.6. *Vertices of a Great Circle.* The vertices of a great circle on the complex plane lie on a line drawn through the centre of the circle, z_c, and the origin. Any point on the line can be generated by scaling z_c by a real constant. From simple geometry it then follows that the vertices of the circle, z_v, lie at the points

$$z_v = z_c \left(1 \pm \frac{r}{|z_c|} \right) = z_c \left(1 \pm \sqrt{1 + |z_c|^{-2}} \right) \tag{15}$$

2.7. *Vertices of a Great Circle passing a Point and Subject to a Limiting Parallel.* Under composite sailing separate great circle tracks are constructed passing through the departure and destination points and tangent to some limiting parallel of latitude. A parallel sailing track is followed between the points of tangency or vertices of the great circles.

It was shown in Stuart (1984) that the radius, r, of a great circle on the complex plane is equal to the secant of its inclination to the equator, ε, which is identical to the latitude of the great circle's vertex. It follows from Equation (15) that the centre, z_c, of the required great circle satisfies the condition $|z_c| = \tan\varepsilon$ and passes through either the point of departure or destination denoted by X which maps to z_X on the complex plane. These requirements along with Equation (5) give

$$z_c = z_1 \left\{ 1 \pm i\sqrt{|z_c|^2 \,/\, |z_1|^2 - 1} \right\} \tag{16}$$

where $z_1 = (|z_X|^2 - 1)/(2\bar{z}_X)$.

Note that there are two possible great circles that satisfy the requirements; one with vertex to the east of z_X and the other to the west. With the mapping of Equation (1) when X and the vertex are in the same hemisphere the upper (lower) sign in Equation (16) corresponds to a vertex to its east (west). This is reversed when they are in different hemispheres.

From Equation (15) and the requirement that $|z_c| = \tan\varepsilon$ the vertex of the great circle on the limiting parallel is

$$z_v = z_c\,(1 \pm \csc\varepsilon) \tag{17}$$

where the upper (lower) sign applies to parallels in the northern (southern) hemisphere.

From Equation (16) it also follows that the difference in longitude between X and the nearer vertex is

$$\mathrm{DLo} = \arg\left(1 + i\sqrt{|z_c|^2 / |z_1|^2 - 1}\right) = \tan^{-1}\sqrt{|z_c|^2 / |z_1|^2 - 1} \tag{18}$$

2.8. *Example.* As shown in Figure 1, a great circle course runs from Yokohama, Japan 35°28′ N, 139°41′ E to San Francisco, USA 37°49′N, 122°25′W. Where does the vertex of this great circle lie? If composite sailing is desired that does not exceed 45°N, what are the beginning and end points of the parallel track?

The positions of Yokohama and San Francisco are mapped to complex numbers z_{YK} and z_{SF} respectively using Equation (1) yielding respectively

$$z_{\mathrm{YK}} = -1{\cdot}479389 + 1{\cdot}255353i; \quad z_{\mathrm{SF}} = -1{\cdot}094663 - 1{\cdot}723804i$$

From Equation (8) the centre of the great circle on the complex plane passing through these two points is

$$z_c = -1{\cdot}114150 - 0{\cdot}211900i$$

The vertex is found from Equation (17) to be

$$z_v = -2{\cdot}599553 - 0{\cdot}494409i$$

which by means of Equation (2) is found to correspond to the position 48°35·8′ N, 169°13·9′W on the sphere.

A composite sailing in which the latitude does not exceed 45° N requires finding a great circle passing Yokohama with a vertex at 45° N and a similar one passing through San Francisco.

Making the substitutions $z_X = z_{\mathrm{YK}}$ and $|z_c| = \tan 45°$ in Equation (16) and selecting the upper sign as is appropriate for a vertex to the east gives the centre of the required great circle passing through Yokohama as

$$z_c = -0{\cdot}997248 - 0{\cdot}074135i$$

By the application of Equation (17) and using the upper sign to select the northern hemisphere vertex

$$z_v = -2{\cdot}407570 - 0{\cdot}178978i$$

corresponding to 45° N, 175°44·9′ W

The analogous calculation applied to San Francisco gives

$$z_c = -0{\cdot}948366 - 0{\cdot}317178i$$

$$z_v = -2{\cdot}289558 - 0{\cdot}765735i$$

with the vertex at 45° N, 161°30·5′ W.

3. GREAT CIRCLE SAILING.

3.1. *The Direct Problem.* The aim is to find the point, z_D, lying at a distance, D, from an initial point, z_1, along a great circle course with initial bearing, C, measured eastward from north. The transformation

$$\mathrm{T}\left(z\right)=\frac{z-z_1}{\bar{z}_1 z+1} \tag{19}$$

describes a rotation of the sphere (Nevanlinna and Paatero, 1969; Stuart, 2009a) that maps the point z_1 to the zero and the north pole, represented by infinity on the complex plane to the point $1/\bar{z}_1$ which defines the direction from which courses are specified. Under this transformation great circles through the point z_1 become straight lines through the origin. A straight line drawn from zero to the point

$$z_2=\tan\frac{D}{2}e^{iC}\frac{z_1}{|z_1|} \tag{20}$$

is a segment of a great circle track from z_1 to a point at a distance of D (radians) and bearing C measured eastward from north. In the original coordinate system prior to rotation this point is

$$z_D=\mathrm{T}^{-1}\left(z_2\right)=z_1\left(\frac{1+|z_1|^{-1}e^{iC}\tan\frac{D}{2}}{1-|z_1|e^{iC}\tan\frac{D}{2}}\right) \tag{21}$$

3.2. *The Inverse Problem.* Conversely the great circle distance D and bearing C of the point z_2 from z_1 is

$$\begin{aligned} D&=2\tan^{-1}\left|\mathrm{T}\left(z_2\right)\right| \\ C&=\arg\left(\bar{z}_1\mathrm{T}\left(z_2\right)\right)=\arg\left(\mathrm{T}\left(z_2\right)\right)-\arg\left(z_1\right) \end{aligned} \tag{22}$$

3.3. *Example.* Find a number of points along a great circle track from latitude 38°N, longitude 125°W when the initial great circle course angle is 291°.

The initial point on the complex plane is $z_1=-1{\cdot}176006-1{\cdot}679511i$ giving

$$|z_1|=2{\cdot}050304$$
$$e^{iC}=-0{\cdot}358368-0{\cdot}933580i$$

Selecting the desired values of the distance, D, and successively using Equation (21) to find z_D and Equation (2) to determine the corresponding latitude and longitude gives

D(n.m.)	*300*	*600*	*900*	*3600*
z_D	*−1·283622–1·488293i*	*−1·354979–1·296712i*	*−1·394770–1·111377i*	*−1·065272–0·016579i*
Latitude	*36·1° N*	*33·9° N*	*31·4° N*	*3·6° N*
Longitude	*130·8° W*	*136·3° W*	*141·5° W*	*179·1° W*

The results agree with those in Bowditch (2002. Section 2407, Example 2).

4. RHUMBLINE SAILING.

A rhumbline or loxodrome is a track of constant heading and therefore cuts each meridian that it crosses at a constant angle before terminating at the poles. When treating an ellipsoidal Earth it is common practice, see for example

Bowditch (2002) to incorporate the eccentricity, e, into expressions for the so-called meridional parts but neglect it when calculating meridional arc lengths. While not strictly correct (Earle, 2005) this practice will be followed here. It is found that this approximation introduces a maximum error of 17 nm or 0·3% of the total meridional arc length. The exact treatment on the ellipsoid by classical methods is given by Williams (1998) and Tseng et al. (2012).

To incorporate the eccentricity define a modification to Equation (1)

$$(L, \lambda) \rightarrow \hat{z} = \left[\frac{1 - e\sin L}{1 + e\sin L}\right]^{\frac{e}{2}} \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) e^{i\lambda} \tag{23}$$

From an operational standpoint the approximation described above amounts to distinguishing between z and $\hat{z}$ only when logarithms are present or equivalently where meridional parts are concerned.

Under stereographic projection meridians are radial lines on the complex plane and the loxodrome with course heading, C, passing through the point z_1 with longitude λ_1 is therefore a logarithmic spiral which as a function of the longitude, λ, takes the form

$$\hat{z} = \hat{z}_1 e^{(\cot C + i)(\lambda - \lambda_1)} \tag{24}$$

4.1. *The Direct Problem.* It is required to find the point z_2 reached by starting at point z_1 and following a rhumb line course with bearing, C, and distance D. Let z_1 and z_2 represent points with latitude and longitudes (L_1, λ_1) and (L_2, λ_2) respectively. The meridional arc length of the track is

$$L_2 - L_1 = D\cos C \tag{25}$$

where the left hand side of Equation (25) is exact on a sphere but is an approximation on an ellipsoid. It follows that

$$|z_2| = \tan\left(\frac{\pi}{4} + \frac{L_1 + D\cos C}{2}\right) = \frac{|z_1| + B}{1 - |z_1|B} \tag{26}$$

where $B = \tan\left(\frac{D\cos C}{2}\right)$ and L_2 can then be obtained using

$$L_2 = 2\tan^{-1}|z_2| - \frac{\pi}{2} \tag{27}$$

Further

$$|\hat{z}_2| = |z_2|\left[\frac{1 - e\sin L_2}{1 + e\sin L_2}\right]^{\frac{e}{2}} \tag{28}$$

Solving Equation (24) for the difference in longitude between z_2 and z_1 yields

$$\lambda_2 - \lambda_1 = \ln\left|\frac{\hat{z}_2}{\hat{z}_1}\right| \tan C \tag{29}$$

and plugging this back into Equation (24) gives the general result

$$\hat{z}_2 = \hat{z}_1 \left|\frac{\hat{z}_2}{\hat{z}_1}\right|^{1 + i\tan C} \tag{30}$$

Within the approximation discussed earlier this is

$$z_2 = z_1 \left|\frac{z_2}{z_1}\right| \left|\frac{\hat{z}_2}{\hat{z}_1}\right|^{i \tan C} = z_1 \left|\frac{z_2}{z_1}\right| \exp\left(i \tan C \ln \left|\frac{\hat{z}_2}{\hat{z}_1}\right|\right) \tag{31}$$

4.2. *The Inverse Problem.* The rhumbline bearing, C, from point z_1 to z_2 follows from simple geometry of their images $\ln \hat{z}_1$ and $\ln \hat{z}_2$ under Mercator projection and is

$$C = \arg \ln \frac{\hat{z}_2}{\hat{z}_1} \tag{32}$$

The rhumbline distance, D, from z_1 to z_2 can be obtained from Equation (25) which may be recast as

$$D = 2\left(\frac{\tan^{-1}|z_2| - \tan^{-1}|z_1|}{\cos C}\right) = 2 \sec C \tan^{-1}\left(\frac{|z_2| - |z_1|}{1 + |z_1|\,|z_2|}\right) \tag{33}$$

4.3. *Example.* A ship at 75°31·7′ N, 79°08·7′ W, steams 263·5 miles on course 155°. Find the latitude and longitude of the point of arrival.

The departure point on the complex plane from Equation (1) is $z_1 = 1{\cdot}483278 - 7{\cdot}735268i$ leading by the analogue of Equation (28) to $\left|\hat{z}_1\right| = 7{\cdot}825203$. Using Equations (26), (27) and (28) with $B = -0{\cdot}034748$ gives

$$L_2 = 71°32{\cdot}9'\mathrm{W}$$
$$\left|\hat{z}_2\right| = 6{\cdot}117479$$

From Equation (31) the point of arrival on the complex plane is $z_2 = 1{\cdot}844419 - 5{\cdot}873752i$ corresponding to 71°32·9′ N, 72°34·0′ W which agrees with Bowditch (2002, Section 2416, Example 2) up to the level of rounding error in longitude present in that reference.

5. CONCLUSIONS.

It has been demonstrated that mapping points on the sphere to points on the complex plane by stereographic projection leads to simple compact solutions to a variety of navigational problems. This work focussed on the problems involving great circles or rhumb lines. The formulae obtained are compact and straightforward to evaluate in practice. Comparison of worked examples to standard references provides verification of the correctness of the expressions obtained.

REFERENCES

Arfken, G.B. and Weber, H.J. (1995). *Mathematical Methods for Physicists*, Academic Press.

Bowditch, N. (2002). *The American Practical Navigator: An Epitome of Navigation*, The National Imagery and Mapping Agency.

Donnay, J.D.H. (1945). *Spherical Trigonometry after the Cesàro Method*, Interscience Publishers, Inc., New York.

Earle, M.A. (2005). A Comment on Navigation Instruction, *The Journal of Navigation,* **58**, 337–340.

Heard, W.B. (2006). *Rigid Body Mechanics*, Wiley-VCH Verlag, Weinheim.

Nevanlinna, R. and Paatero, V. (1969). *Introduction to Complex Analysis*, Reading Massachusetts.

Stuart, R.G. (1984). *Applications of Complex Analysis to Spherical Coordinate Geometry*, Quarterly Journal of the Royal Astronomical Society **25**, 126–136; http://articles.adsabs.harvard.edu/full/1984QJRAS..25..126S

Stuart, R.G. (2009a). Applications of Complex Analysis to Celestial Navigation, *NAVIGATION: Journal of the Institute of Navigation*, **56,** 221–227.
Stuart, R.G. (2009b). Applications of complex analysis to precession, nutation and aberration, *Monthly Notices of the Royal Astronomical Society,* **400,** 1366–1372.
Tseng, W.-K., A. Earle, M.A. and Guo, J.-L. (2012). Direct and Inverse Solutions with Geodetic Latitude in Terms of Longitude for Rhumb Line Sailing. *The Journal of Navigation*, **65**, 549–559.
Williams, R. (1998). *Geometry of Navigation*, Horwood Publishing Ltd, Chichester, UK.

APPENDIX. LINE OF POSITION NAVIGATION.

LoP navigation involves comparing the observed altitude of a celestial body to a calculated altitude for a reference or Assumed Position (AP) at latitude, λ, and longitude, L. Noting that the body's Greenwich Hour Angle, GHA, and declination, δ, are related to its altitude, h, and azimuth, Z, by a coordinate rotation, Stuart (2009a) derived expressions for the latter in the form of a bilinear or Möbius transformation

$$\mathrm{T}(z) = \frac{az + b}{-\bar{b}z + \bar{a}} \tag{A1}$$

of points on the complex plane. This result can be written intuitively and succinctly as follows. Let z_{GP} be the image on the complex plane of the geographic position of the celestial body and z_{AP} be the image of the observer's assumed position

$$z_{\mathrm{GP}} = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right) e^{-i\,(\mathrm{GHA})}; \quad z_{\mathrm{AP}} = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right) e^{i\,\lambda}.$$

The required altitude and azimuth can be found from

$$\tan\left(\frac{\pi}{4} - \frac{h}{2}\right) e^{iZ} = \mathrm{T}(z_{\mathrm{GP}}) = e^{-i\lambda}\left(\frac{z_{\mathrm{GP}} - z_{\mathrm{AP}}}{\bar{z}_{\mathrm{AP}} z_{\mathrm{GP}} + 1}\right) \tag{A2}$$

and hence

$$h = \frac{\pi}{2} - 2\tan^{-1}|\mathrm{T}(z_{\mathrm{GP}})|; \quad Z = \arg \mathrm{T}(z_{\mathrm{GP}}) \tag{A3}$$

Alternatively in terms of the three quantities, Local Hour Angle, LHA, latitude, L, and declination, δ, normally used as input into LoP calculations, this may be written

$$\tan\left(\frac{\pi}{4} - \frac{h}{2}\right) e^{iZ} = \mathrm{T}(z_{\mathrm{GP}}) = \frac{|z_{\mathrm{GP}}|\, e^{-i(\mathrm{LHA})} - |z_{\mathrm{AP}}|}{|z_{\mathrm{GP}}|\,|z_{\mathrm{AP}}|\, e^{-i(\mathrm{LHA})} + 1} \tag{A4}$$

where

$$|z_{\mathrm{GP}}| = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right); \quad |z_{\mathrm{AP}}| = \tan\left(\frac{\pi}{4} + \frac{L}{2}\right)$$

and

$$\mathrm{LHA} = \mathrm{GHA} + \lambda.$$