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Vector Solution for the Intersection of Two Circles of Equal Altitude

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A direct method for obtaining the two possible positions derived from two sights using vector analysis instead of spherical trigonometry is presented. The geometry of the circle of equal altitude and of the two body fixes is analyzed, and the vector equation for simultaneous sights is constructed. The running fix problem is also treated.

KEY WORDS

1. Circle of Equal Altitude. 2. Celestial Navigation. 3. Sight Reduction. 4. Vector Analysis.

1. INTRODUCTION. Most of the known methods to solve the two body problem, [10], [13–17], [19], [24–26], use the navigational triangle [1] and spherical trigonometry. Recursive methods using linear approximations as in [18], or physics analogies [12] are smart approaches. Sight reduction general procedures [2–5], can be used to obtain the fix, but for two sights the estimated position is needed. In reference [11] an analytical solution using plane geometry is used for both solutions.

On the other hand, vector calculus is a powerful tool widely used for engineering and physics, [9]. In navigation it is used in naval kinematics, for current calculations, (set & drift) and the great circle sailing can be formulated using the vector analysis, [20–23]; in [27] a vector approach of celestial navigation is made. In this paper a solution for the two body problem in celestial navigation using this technique is presented. The following sub paragraphs present some concepts about the celestial circle of position, *CoP*.

1.1. *Variables and symbols.* Let us denote the dot product by • and the cross product by Λ, and a vector by $\vec{v}$ or by $\mathbf{v}$. The variables used are:

	Variable	Intervals
GHA	Greenwich Hour Angle	$0 <= \text{GHA} <= 360^\circ$ (W to E)
Dec	Declination	-90° (S) $<= \text{Dec} <= +90^\circ$ (N)
Ho	Observed altitude	$0 <= \text{H} <= 90^\circ$
B	Latitude	-90° (S) $<= \text{B} <= +90^\circ$ (N)
L	Longitude	-180° (W) $<= \text{L} <= +180^\circ$ (E)

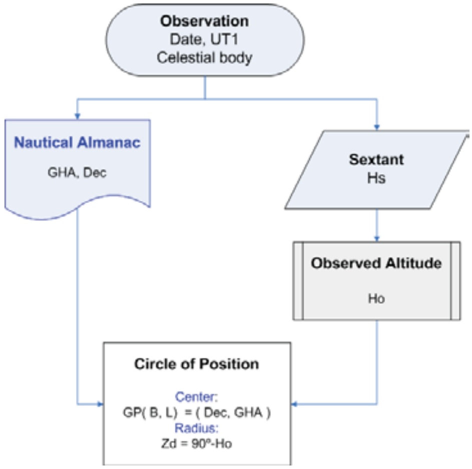

Figure 1. Circle of Equal Altitude parameters.

1.2. *The Circle of Equal Altitude.* A celestial object is far enough away from the observer that the incoming light rays are nearly parallel to each other. Thus, there is a point on the surface of Earth where the object is directly overhead at a given time; this point is called the *geographical position*, *GP*, (*or of a star; substellar point*). Using the spherical model of the Earth, there is a circle on her surface centred about the object's geographical position where the angle between the horizon and the celestial object, called the altitude, is constant at a given instant. This circumference forms a celestial line of position, a small circle, known as a circle of equal altitude. The great circle distance from this pole to the circle is the zenith distance of the body, Zd.

At the time of observation the observer of the celestial object must be located somewhere along that circle. The geographical position of a celestial body is calculated from an ephemeris or obtained from the *Nautical Almanac*, and the altitude is measured with a sextant. The observed altitude is the sextant altitude corrected for index error and dip, for refraction and if appropriate corrected for parallax and semi-diameter. This process is summarized in Figure 1.

Geometrically a Circle of Equal Altitude is generated by the intersection of a circular cone having its vertex in the centre of the Earth, half angle $\alpha = 90°$-Ho, and with the vector from O to GP as its axis, with the unit sphere. The distance from the centre of the Earth to the plane containing the CoP is $sin(Ho)$, as shown in Figure 2.

1.3. *Vector equation for the Circle of Equal Altitude.* In Figure 3, let OP be the observer's position at the time of sight, and GP the geographical position of the celestial body at the same instant. The dot product of the vectors defined by the centre of the Earth and these points is the cosine of the angle between them, which is the zenith distance of the observed body. Then, the vector equation of the circle of equal altitude is:

$$\vec{O}P \bullet \vec{G}P = cos(90° - Ho) \tag{1}$$

It is possible to write the azimuth in vector form [8].

Figure 2. Earth's normal section to the plane of the Circle of Position (CoP).

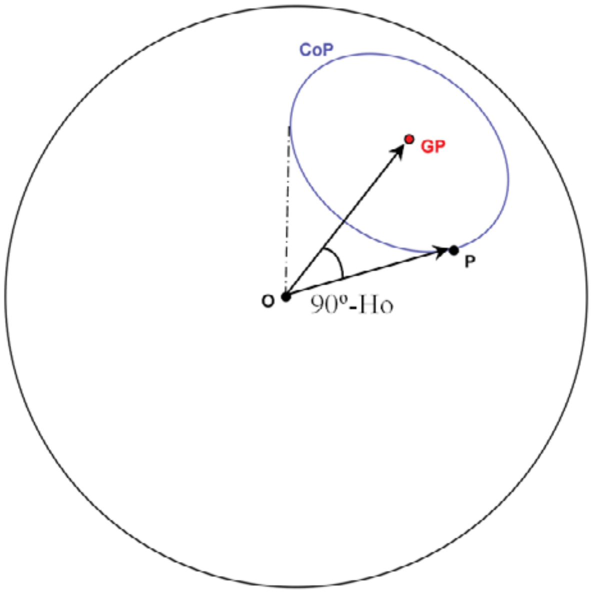

Figure 3. Circle of Equal Altitude and vectors.

1.4. *Coordinate Systems*. Using a right-handed orthonormal basis $\{\vec{i}, \vec{j}, \vec{k}\}$, as in Figure 4, the Cartesian system of coordinates is defined where the origin **O**, is the centre of the Earth with Axes:

- Z: from O to the North Pole.
- X: from O to the Greenwich meridian, included in the Earth's equatorial plane.
- Y: defined by $\vec{j} = \vec{k} \wedge \vec{i}$

Since angles are used, not distances, the hypothesis that the Earth is a sphere of unit radius is valid.

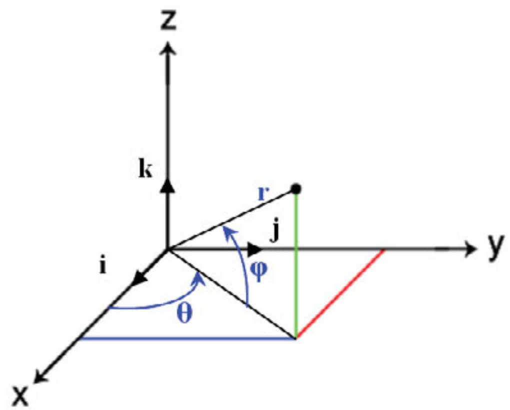

Figure 4. Right-handed orthonormal basis, and coordinates.

The relationship between the equatorial coordinates (Dec, GHA), and geographical coordinates (B, L), with the spherical ones, (φ, θ), arises from Figure 4:

$$\begin{array}{ll} r=1 & r=1 \\ Dec=\varphi & B=\varphi \\ GHA=360^\circ\text{-}\theta & L=\theta \end{array}$$

According to this formulation the unit vector in Cartesian coordinates, (x,y,z), from the centre of the Earth to the geographical position of any body is:

$$\overrightarrow{\mathbf{GP}} = cos\,Dec\,cos\,GHA \cdot \vec{i} - cos\,Dec\,sin\,GHA \cdot \vec{j} + sin\,Dec \cdot \vec{k} \tag{2}$$

And the unit vector in Cartesian coordinates from the centre of the Earth to any point on the surface of the Earth is:

$$\overrightarrow{\mathbf{OP}} = cos\,B\,cos\,L \cdot \vec{i} + cos\,B\,sin\,L \cdot \vec{j} + sin\,B \cdot \vec{k} \tag{3}$$

2. VECTOR EQUATION FOR THE INTERSECTION OF TWO SIMULTANEOUS CIRCLES OF POSITION. In the general case, two CoP intersect at two points: I_1 and I_2, (Figure 5). The coordinates of these two crossings are the solution to the problem. Using the vector notation in section 1.3 there are three unknown variables, the Cartesian coordinates of **OP;** so three equations are needed:

$$\begin{aligned} \overrightarrow{\mathbf{OP}} \bullet \overrightarrow{\mathbf{GP}}_1 &= sin\,Ho_1 \\ \overrightarrow{\mathbf{OP}} \bullet \overrightarrow{\mathbf{GP}}_2 &= sin\,Ho_2 \\ \overrightarrow{\mathbf{OP}} \bullet \overrightarrow{\mathbf{OP}} &= 1 \end{aligned} \tag{4}$$

The last equation takes account of the fact that the observer is on the surface of a unit sphere: $x^2+y^2+z^2=1$. To solve the system some methods have been published [4], [11].