A comparison of the two curves shows that maximum altitude occurs about five minutes after meridian altitude.

Formula Giving the Interval Between Meridian Passage and Maximum Altitude. It is apparent that, in the circumstances described above, maximum altitude occurs at the moment when the rate of increase in altitude resulting from one factor or set of factors is exactly equal to the rate of decrease in altitude resulting from another factor or set of factors.

In figure 73, Z is the observer's zenith when the Sun is on the meridian at X, and X' is the Sun's position at maximum altitude. The observer's zenith is then Z', the angle PZZ' being (360° — course). The angle $PZ'X'$ is denoted by α, and the latitude of Z' is l. Then, since the north–south component of the ship's movement during

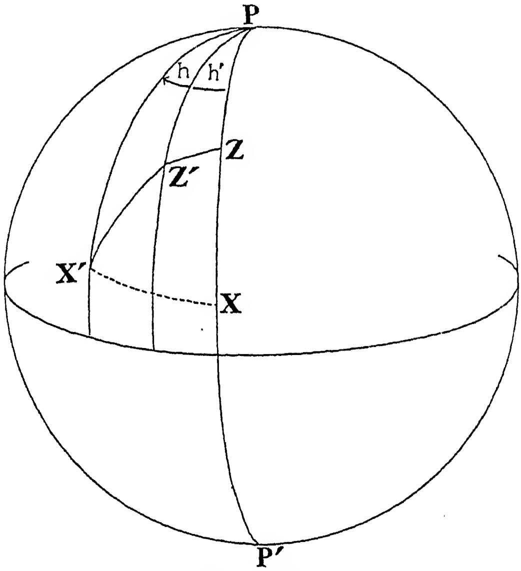

FIGURE 73.

the short interval between meridian and maximum altitude is small compared with the distance travelled by the Sun—that is, ZZ' is negligible in comparison with XX'—the latitude of Z may be taken as the latitude of Z'.

From the spherical triangle $PZ'X'$, in which h denotes the hour angle of the Sun at the ship when the altitude is a maximum :

$$\sin \alpha = \frac{\sin h \cos d}{\sin Z'X'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let x denote the ship's rate of change in longitude, and y the rate of north–south change in latitude combined with the rate of change in declination, both x and y being expressed in minutes of arc per hour. If the north–south component of the ship's movement and the change in declination have opposite names, y is the sum of the rates of change in latitude and declination ; if they have the same names, it is the difference.

When the ship is stationary, the rate of change in altitude resulting from the Earth's rotation is—see page 145—given by :

$$15 \cos l \sin \alpha \quad \text{in minutes of arc per minute of time}$$

i.e. $$900 \cos l \sin \alpha \quad \text{in minutes of arc per hour}$$

When the ship's change in longitude is taken into account, this rate of change becomes :

$$(900 \mp x) \cos l \sin \alpha$$

—the upper sign being taken when the ship is moving west, and the lower when she is moving east.

Maximum altitude occurs when the two rates of change in altitude resulting from :

(1) the Earth's rotation combined with the ship's rate of change in longitude.

(2) the ship's north–south speed combined with the Sun's rate of change in declination.

—are equal but of opposite sign. Hence :

$$y = (900 \mp x) \cos l \sin \alpha$$

i.e. $$\sin \alpha = \frac{y \sec l}{900 \mp x}$$

Therefore, when this value of $\sin \alpha$ is substituted in equation (5) and $(l \pm d)$ is written for $Z'X'$:

$$\sin h = \frac{y \sin (l \pm d)}{(900 \mp x) \cos l \cos d}$$

$$= \frac{y (\tan l \pm \tan d)}{900 \left[1 \mp \dfrac{x}{900} \right]}$$

But h is small, and if it is expressed in seconds of time, four seconds of time being equal to one minute of arc :

$$\sin h = \frac{h}{4 \times 3{,}438}$$

Also, since $\dfrac{x}{900}$ is small :

$$\frac{1}{1 \mp \dfrac{x}{900}} = 1 \pm \frac{x}{900}$$

Hence : $$h = \frac{4 \times 3{,}438}{900} y \left[1 \pm \frac{x}{900} \right] (\tan l \pm \tan d)$$

—the upper signs being taken when the ship is moving west and the latitude and declination have opposite names, and the lower when she is moving east and the latitude and declination have the same names.

In figure 73, the interval between meridian passage and maximum altitude is represented by the angle ZPZ'. If this angle is denoted by h' and the angle ZPX' by H, then:

$$H = h \pm h'$$

But h' is the d'long through which the ship moves in the interval H. Therefore:

$$\frac{h'}{H} = \frac{\text{rate at which the ship is changing her longitude}}{\text{rate at which the Sun is changing its longitude}}$$

$$= \frac{x}{900}$$

i.e.
$$h' = \frac{Hx}{900}$$

Hence:
$$h = H \mp h' = H\left[1 \mp \frac{x}{900}\right]$$

i.e.
$$= h\left[1 \pm \frac{x}{900}\right]$$

—the upper sign being taken when the ship is moving west and the lower when she is moving east.

By substitution, therefore:

$$H = \frac{4 \times 3{,}438}{900}\, y \left[1 \pm \frac{x}{900}\right]^2 (\tan l \pm \tan d)$$

i.e.
$$H = 15{\cdot}28\, y \left[1 \pm \frac{2x}{900}\right] (\tan l \pm \tan d)$$

—the upper signs being taken when the ship is moving west and the latitude and declination have opposite names, and the lower when she is moving east and the latitude and declination have the same names.

The quantities $15{\cdot}28 \tan l$ and $15{\cdot}28 \tan d$ can be found from the following table, and the value of $15{\cdot}28 (\tan l \pm \tan d)$ obtained by addition or subtraction.

TABLE GIVING $15{\cdot}28$ TAN l AND $15{\cdot}28$ TAN d.

l or d		l or d		l or d		l or d	
°		°		°		°	
0	0·0	16	4·4	31	9·2	46	15·8
1	0·3	17	4·7	32	9·5	47	16·4
2	0·5	18	5·0	33	9·9	48	17·0
3	0·8	19	5·3	34	10·3	49	17·6
4	1·1	20	5·6	35	10·7	50	18·2
5	1·3	21	5·9	36	11·1	51	18·9
6	1·6	22	6·2	37	11·5	52	19·5
7	1·9	23	6·5	38	11·9	53	20·3
8	2·1	24	6·8	39	12·4	54	21·0
9	2·4	25	7·1	40	12·8	55	21·8
10	2·7	26	7·4	41	13·3	56	22·7
11	3·0	27	7·8	42	13·8	57	23·5
12	3·2	28	8·1	43	14·3	58	24·5
13	3·5	29	8·5	44	14·8	59	25·4
14	3·8	30	8·8	45	15·3	60	26·5
15	4·1	—	—	—	—	—	—

Required the interval between meridian passage and maximum altitude in 40°N., 60°W. on the 3rd April 1937 when the observer is in a ship steaming 230° at 16 knots.

By traverse table for a run of one hour :

$$\begin{aligned} \text{d'lat} &= 10'{\cdot}3\text{S.} \\ \text{dep.} &= 12'{\cdot}3\text{W.} \\ \therefore \qquad \text{d'long} &= 16'{\cdot}1\text{W.} \end{aligned}$$

Meridian passage in 60°W. occurs at about $16^{\text{h}}00^{\text{m}}$ G.M.T., and at this hour the Sun's declination is seen, from the *Nautical Almanac*, to be 5°19′N., increasing by 1′ per hour. Therefore x is equal to 16·1 and y to 11·3, and :

$$\begin{aligned} & 15{\cdot}28\,(\tan l - \tan d) \\ &= 12{\cdot}8 - 1{\cdot}4 \\ &= 11{\cdot}4 \end{aligned}$$

The interval between meridian passage and maximum altitude is thus :

$$\begin{aligned} & 11{\cdot}3\left(1+\frac{2\times 16{\cdot}1}{900}\right)11{\cdot}4 \\ &= 133^{\text{s}} \\ &= 2^{\text{m}}13^{\text{s}} \end{aligned}$$

Alternative Proof. The formula giving the interval between meridian passage and maximum altitude can be obtained by direct differentiation of the fundamental formula with respect to the time t. Thus, in the usual notation :

$$\cos z = \cos p \cos c + \sin p \sin c \cos h$$

$$\begin{aligned} \therefore \qquad -\sin z\frac{dz}{dt} &= (\cos p \sin c \cos h - \sin p \cos c)\frac{dp}{dt} \\ &\quad + (\sin p \cos c \cos h - \sin c \cos p)\frac{dc}{dt} \\ &\quad - \sin p \sin c \sin h\frac{dh}{dt} \qquad \ldots\ldots \quad (6) \end{aligned}$$

When the altitude is a maximum, z is a minimum, so that $\frac{dz}{dt}$ is zero.

Also h is small. Therefore no appreciable error will occur if $\cos h$ is taken as unity, and the polar distance and co-latitude at meridian passage are used. Formula (6) then reduces to :

$$\sin h\,\frac{dh}{dt} = \frac{\sin\,(p-c)}{\sin p \sin c}\left[\frac{dc}{dt} - \frac{dp}{dt}\right]$$

But $\frac{dh}{dt}$ is the rate at which the Sun's hour angle changes.

Therefore : $$\frac{dh}{dt} = 900 \pm x$$

Also $\left[\frac{dc}{dt}-\frac{dp}{dt}\right]$ is the rate of change of $(c-p)$.

Therefore:
$$\frac{dc}{dt}-\frac{dp}{dt}=y$$

Formula (6) thus becomes:
$$\sin h=\frac{y \sin (l-d)}{(900 \pm x) \cos l \cos d}$$

The remainder of the work is the same as that on page 153.

Longitude by Equal Altitudes. If T_1 and T_2 are the times shown by the chronometer when a heavenly body (of constant declination) has the same altitudes before and after passage across a stationary observer's meridian, it is evident from figure 69 that meridian passage must take place at a time $\frac{1}{2}(T_1+T_2)$ by the chronometer. Hence the G.M.T of meridian passage can be found and from this the longitude of the ship at meridian passage, since the longitude is simply the difference between this G.M.T. and the L.M.T. obtained by subtracting the astronomical quantity E from an hour angle of $0^h0^m0^s$.

When the movement of the ship and the change of declination are taken into account, the mean of the chronometer times, $\frac{1}{2}(T_1+T_2)$ is approximately the time of maximum altitude. It would be the exact time if the curve in figure 72 were symmetrical, but the error introduced by regarding it as symmetrical has no practical importance if the altitudes are taken within about half an hour of the time of meridian passage. Then:
$$H=15{\cdot}28y\left(1 \pm \frac{2x}{900}\right)[\tan l \pm \tan d]$$
—and this quantity, when applied to $\frac{1}{2}(T_1+T_2)$, gives the time of meridian passage and therefore the ship's longitude.

In order that the times of equal altitudes may be observed accurately, the altitude of the heavenly body must be changing appreciably when the observation is taken. The heavenly body must not, therefore, be too close to the meridian. The formula giving the rate of change in altitude—$15'$ cos l sin (*az.*) per minute—shows that this rate is only $0'{\cdot}1$ per second in latitude 30° when the azimuth is 150°. In higher latitudes, or when the heavenly body is nearer the meridian, it is less. The limiting azimuth, consistent with accuracy in sight-taking, is thus about 160°.

If the limiting time-interval from the meridian is now taken as 40^m, the limiting rate of change in azimuth can be found from formula (2). This rate in minutes of arc per minute of time is:
$$15 \sin (az.) \operatorname{cosec} h \cos \beta$$
—and, since β is approximately 0, the rate in the limiting circumstances is therefore:
$$15 \sin 160° \operatorname{cosec} 40^m$$
—or at least 30° per hour.