NAVIGATION: *Journal of The Institute of Navigation*
Vol. 38, No. 1, Spring 1991
Printed in U.S.A.

On the Overdetermined Celestial Fix

THOMAS R. METCALF
Institute for Astronomy
University of Hawaii, Honolulu, Hawaii

FREDERIC T. METCALF
Department of Mathematics, University of California,
Riverside, California
Received June 1990
Revised December 1990

ABSTRACT

We present an algorithm for computing the best position fix (in the least-squares sense) from a sequence of observations of any number of celestial bodies. This algorithm has several advantages, most notably that it does not rely on an initial guess of the position, is computationally expedient, and is simple enough to program on the more sophisticated hand-held calculators. We present the mathematical background for this algorithm, a description of its implementation, examples of its use on both simulated and "real" data, and an estimate of the error on the derived position.

INTRODUCTION

In a recent paper [1], a method was presented for determining a position fix from a relatively large number of rapid observations of the altitude of one or more celestial bodies using the method of least-squares fitting. While this method is quite useful, we have derived an alternative algorithm for performing the least-squares fit. This algorithm has several advantages over the method presented in [1]. First, it is more expedient computationally and is simple enough to be readily programmed on the growing number of sophisticated hand-held calculators. Second, and most significant, our algorithm does not require an initial guess for the position and hence will not converge to an incorrect solution; least-squares algorithms that proceed from some initial value by "sliding" down the gradient of the goodness-of-fit parameter can converge to a local minimum that is not the "best" solution.

The central idea behind the overdetermined fix is to find the best fit to the altitude of a celestial body observed as a function of time. Since the altitude is a known function of the position, this best fit yields optimal values of latitude and longitude. Moreover, a large number of observations can be used in the fit; random errors in the observations will tend to cancel out, yielding a fix that can be substantially more accurate than might be expected given the magnitude of the errors on the altitude observations and the short period of time spanned by the observations. In fact, using these methods, an accurate fix can quite often be determined from 10 to 20 observations of a single celestial body spanning only 10 to 20 min of time.

As discussed in [1], obtaining an accurate position fix from observations of a single body when its azimuth is changing slowly requires a longer sequence of observations. This condition occurs primarily when the body is rising or setting, and the observer's latitude is in the tropics. However, it also occurs in the tropics when the body's declination is close to the observer's latitude, since the azimuth of the celestial body changes very slowly before transit, rapidly during transit, and again slowly after transit. This can be compensated for by either including observations both before and after transit, or including observations spanning 30 min or more (see equation (3)). In the worst case, if the body's declination and the observer's latitude are both close to zero, a very long sequence of observations may be required for an accurate fix. See [2] for a good discussion of the motion of celestial bodies.

In an earlier work ([3]; see also [4, p. 118]), the method of least-squares fitting was applied to sightings of separate bodies. The method presented here also allows observations of separate bodies; moreover, since it is correct mathematically, it does not suffer from the inaccuracy that is introduced in the method of [3] when the normalization $X^2 + Y^2 + Z^2 = 1$ (the notation of [3]) is ignored in the least-squares problem (see the mathematical discussion below and the introduction of a Lagrange multiplier to handle this normalization correctly). It is not possible to correct this error by simply dividing the solution values by $X^2 + Y^2 + Z^2$, as was done in [3].

BACKGROUND

The altitude (H_0) of a celestial body is given as a function of the declination of the body (d, north positive), the Greenwich hour angle (GHA) of the body, the observer's longitude (λ, west positive), and the observer's latitude (L, north positive) by

$$\sin(H_0) = \sin(d)\sin(L) + \cos(d)\cos(L)\cos[\mathrm{GHA} - \lambda]. \tag{1}$$

In this paper, we present an algorithm that finds the best fit to this function, yielding λ and L.

The GHA and the declination can be found from the time of the observation using a simple linear interpolation of values in the *Nautical Almanac* at the whole hours nearest the observation time:

$$\mathrm{GHA} = \mathrm{GHA}_0 + \left[\frac{\mathrm{GHA}_1 - \mathrm{GHA}_0}{T_1 - T_0}\right](t - T_0) \tag{2}$$

where GHA_0 is the GHA at the whole hour T_0, GHA_1 is the GHA at the whole hour T_1, and t is the time of the observation (in hours). If the GHA passes through zero between T_0 and T_1, 360 deg should be added to GHA_1. For the declination, replace GHA by d throughout equation (2). The times T_0, T_1, and t must be measured in the same time zone, and it is assumed that H_0 in equation (1) has been corrected for index, dip, refraction, semidiameter, and parallax, as appropriate. See [4] (p. 117) or [5] (p. B13) for a correction to the observed altitude that accounts for the motion of the observer. Note that [4] and [5] also give a correction for the motion of the observed body. This second correction should *not* be applied since it forces the observations to be simultaneous, contradicting our effort to fit to the motion of the body in time.

In the least-squares fit, at least three observations are required to specify the fix uniquely. However, as pointed out in [3], the observer's dead-reckoning position can be incorporated as "data" simply by adding one observation with H_0 equal to 90 deg, GHA equal to the dead-reckoning longitude, and d equal to the dead-reckoning latitude. Hence, if an accurate dead-reckoning position is known, only two observations are in fact required.

If all observations are of the same body and all are obtained very close to the time of transit, the data can represent equally well observations from the actual latitude or a latitude reflected over the declination of the observed celestial body ($L' = 2d - L$). In this special case, although the longitude is unaffected, the latitude may take on either of two values (L, L'), and the actual latitude must be selected from these using the azimuth of the body at transit (north or south), or the dead-reckoning position. This ambiguity at transit is a general feature of any method that fits to the altitude of a single celestial body since the altitude will peak at the same value at both L and L'.

The errors on the derived position can be calculated using the covariance matrix contained in [1]. However, for observations of a single body, a more practical guide to the errors is obtained from a consideration of the length of time the observations span and the rate of change of the azimuth (see Appendix B). To allow the azimuth to change enough for a good fix, the time interval which the observations span should be chosen to be at least

$$\Delta t \approx \left(\frac{\delta H}{\epsilon\sqrt{n-1}}\right)\frac{57.3\cos(\overline{H})}{\sqrt{225\cos^2(d) - (dH/dt)^2}} \tag{3}$$

where Δt is measured in hours, ϵ is the desired accuracy in miles, δH is the accuracy of the observations in arcminutes, n is the number of observations, $\overline{H}$ is a rough measure of the mean observed altitude, d is the declination of the observed body, and dH/dt is the rate of change of the observed altitude with respect to time (deg/h). Since equation (3) is a guide only, it is sufficient to take the derivative to be

$$\frac{dH}{dt} \approx \frac{H_{0_n} - H_{0_1}}{t_n - t_1}$$

where H_{0_1} and H_{0_n} are the altitudes (degrees) of the first and the last observation, respectively, and t_1 and t_n are the times of these observations (hours). Note that equation (3) can be solved for ϵ to give the estimated error on a set of observations *of a single celestial body*.

ALGORITHM

This description of the algorithm summarizes the rigorous development in Appendix A. The algorithm uses a Lagrange multiplier (μ) to normalize the least-squares problem correctly, and it is the computation of this number which is at the heart of the technique. The algorithm involves the simultaneous solution of a nonlinear equation for μ and three linear equations relating to the position fix. To solve these equations efficiently, we cast the least-squares problem in matrix form and use the eigenvalues (α_i) and eigenvectors ($\mathbf{e}_i$) of the resulting matrix ($\mathbf{G}$). Since the computation of μ involves the solution of a nonlinear equation, μ is found iteratively; once the solution to the four simultaneous equations is known, the position fix is easily computed.

Where possible, the calculations are indicated in a fashion that reduces the number of arithmetic operations. Quantities that are used more than once are placed in parentheses whenever possible. All summations below run over the set of n observations.

Basic Observational Sums:

$$G_{11} = \sum_{i=1}^{n} \sin^2(d_i) \qquad r_1 = \sum_{i=1}^{n} \sin(d_i)\sin(H_{0_i})$$

$$G_{12} = \sum_{i=1}^{n} \sin(d_i)\cos(d_i)\cos(GHA_i) \qquad r_2 = \sum_{i=1}^{n} \cos(d_i)\cos(GHA_i)\sin(H_{0_i})$$

$$G_{13} = \sum_{i=1}^{n} \sin(d_i)\cos(d_i)\sin(GHA_i) \qquad r_3 = \sum_{i=1}^{n} \cos(d_i)\sin(GHA_i)\sin(H_{0_i})$$

$$G_{22} = \sum_{i=1}^{n} \cos^2(d_i)\cos^2(GHA_i)$$

$$G_{23} = \sum_{i=1}^{n} \cos^2(d_i)\sin(GHA_i)\cos(GHA_i) \quad G_{33} = n - G_{11} - G_{22}$$

Eigenvalues:

$$a_1 = (G_{11}G_{22} - G_{12}^2) + (G_{11}G_{33} - G_{13}^2) + (G_{22}G_{33} - G_{23}^2)$$

$$a_0 = -(G_{12}G_{23} - G_{13}G_{22})G_{13} - (G_{13}G_{12} - G_{11}G_{23})G_{23} - (G_{11}G_{22} - G_{12}^2)G_{33}$$

$$Q = \left(\frac{n}{3}\right)^2 - \left(\frac{a_1}{3}\right), \quad R = \frac{a_0}{2} + \left(\frac{n}{3}\right)\left[\left(\frac{a_1}{6}\right) - Q\right], \quad \theta = \arccos\left(R\Big/\left(Q\sqrt{Q}\right)\right)$$

$$\alpha_1 = \left(\frac{n}{3}\right) - \left(2\sqrt{Q}\right)\cos\left(\frac{\theta}{3}\right)$$

$$\alpha_3 = \left(\frac{n}{3}\right) - \left(2\sqrt{Q}\right)\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)$$

$$\alpha_2 = n - \alpha_3 - \alpha_1 \tag{4}$$

These three values have been indexed so that $\alpha_1 < \alpha_2 < \alpha_3$. Note that θ is assumed to be in radians in equation (4), and that the principal value of θ and the positive root of $\sqrt{Q}$ are assumed.

Eigenvectors:

$$\mathbf{e}_i = \begin{pmatrix} (G_{12}G_{23} - G_{13}G_{22}) + G_{13}\alpha_i \\ (G_{13}G_{12} - G_{11}G_{23}) + G_{23}\alpha_i \\ (G_{11}G_{22} - G_{12}^2) - (G_{11} + G_{22})\alpha_i + \alpha_i^2 \end{pmatrix} \quad i = 1, 2, 3 \tag{5}$$

Iterative Determination of μ:

$$\beta_i = \frac{\mathbf{r} \cdot \mathbf{e}_i}{\|\mathbf{e}_i\|} \quad (i = 1, 2, 3) \tag{6}$$

$$g(\mu) = \left(\frac{\beta_1}{\alpha_1 - \mu}\right)^2 + \left(\frac{\beta_2}{\alpha_2 - \mu}\right)^2 + \left(\frac{\beta_3}{\alpha_3 - \mu}\right)^2 - 1 \tag{7}$$

$$g'(\mu) = \frac{2}{\alpha_1 - \mu}\left(\frac{\beta_1}{\alpha_1 - \mu}\right)^2 + \frac{2}{\alpha_2 - \mu}\left(\frac{\beta_2}{\alpha_2 - \mu}\right)^2 + \frac{2}{\alpha_3 - \mu}\left(\frac{\beta_3}{\alpha_3 - \mu}\right)^2 \tag{8}$$

$$\mu_{k+1} = \mu_k - \frac{g(\mu_k)}{g'(\mu_k)}, \text{ with } \mu_0 = 0 \tag{9}$$

Note that the values of α_i and β_i in equations (7) and (8) are computed only once, and then used throughout the iterative process. We have found that terminating the iteration when

$$\left|\frac{(\mu_k - \mu_{k-1})}{\mu_k}\right| < 10^{-6} \tag{10}$$

works well.

Solution for u, v, w:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\|\mathbf{e}_1\|}\left(\frac{\beta_1}{\alpha_1 - \mu}\right)\mathbf{e}_1 + \frac{1}{\|\mathbf{e}_2\|}\left(\frac{\beta_2}{\alpha_2 - \mu}\right)\mathbf{e}_2 + \frac{1}{\|\mathbf{e}_3\|}\left(\frac{\beta_3}{\alpha_3 - \mu}\right)\mathbf{e}_3 \tag{11}$$

Solution for L, λ:

$$L = \arcsin(u) \text{ and } \lambda = \arctan\left(\frac{w}{v}\right) + \begin{cases} +180°, \text{ if } v < 0 \text{ and } w > 0 \\ -180°, \text{ if } v < 0 \text{ and } w < 0 \end{cases} \tag{12}$$

where it is assumed the principal value of the inverse tangent is used.

APPLICATION TO SAMPLE DATA

We have applied this algorithm to two sets of sample data. The first is simulated data for a true position of 21° 12′ N and 157° 30′ W. The altitudes were rounded to the nearest arcminute to simulate errors, and the time of each data point was taken to be exact. The data simulate three observations each of the stars Deneb, Fomalhaut, and Aldebaran, spanning a 12 min time interval beginning at 17:00 Hawaiian Standard Time (HST) (10 h behind UT) on January 1, 1990. The GHA and declinations of the stars were taken from the *Nautical Almanac*. Table 1 shows the data used, which should be sufficient to test any new implementation of the above algorithm. The column labeled Actual Error gives the error in the computed position when the data below the row containing the error value are not used: as more data are included in the fit, the trend towards higher accuracy is clear.

Table 1—Simulated Sightings at (157° 30′ W; 21° 12′ N)

Star	Time (HST)	Altitude (deg:min)	Actual Error (nmi)
Deneb	17:00:00	50:15	
Fomalhaut	17:01:00	38:54	
Aldebaran	17:02:00	15:32	0.28
Deneb	17:05:00	49:27	0.36
Fomalhaut	17:06:00	38:46	0.27
Aldebaran	17:07:00	16:41	0.15
Deneb	17:10:00	48:38	0.18
Fomalhaut	17:11:00	38:37	0.18
Aldebaran	17:12:00	17:50	0.05

The second application of the method is to real observations of the Moon taken on land with an inexpensive plastic sextant, and with a pan of vegetable oil serving as an artificial horizon. The true location was 157° 47.6′ W and 21° 16.2′ N, and the observations consisted of 18 sightings between 22:02 HST and 22:38 HST on June 2, 1990. The results of this test are summarized in Table 2, and illustrate the level of accuracy that can be obtained using only a fair source of data. The column labeled Predicted Error gives the error predicted from equation (3) using an accuracy of 2 arcmin for δH. Using the full data set, the position was found to be 157° 45′ W and 21° 16′ N, an error of about 2 nmi.

SUMMARY

We have presented an algorithm for computing latitude and longitude from a sequence of sextant observations. This method is both computationally less

Table 2—Actual Moon Sightings at (157° 47.6′ W; 21° 16.2′ N)

Time (HST)	Altitude (deg:min)	Actual Error (nmi)	Predicted Error (nmi)
22:02:24	51:49.3		
22:06:35	51:17.2		
22:08:56	50:57.9	51.2	37.4
22:10:32	50:44.6	41.5	24.8
22:15:27	50:02.1	18.1	13.5
22:17:02	49:50.7	28.1	10.8
22:18:09	49:35.9	6.2	9.3
22:19:07	49:30.9	10.7	8.2
22:20:20	49:16.6	2.8	7.2
22:21:36	49:08.7	3.4	6.3
22:25:03	48:34.6	3.9	5.0
22:25:53	48:27.7	5.4	4.6
22:26:50	48:21.2	1.1	4.3
22:34:42	47:03.6	5.5	3.1
22:35:35	46:56.2	5.5	2.9
22:36:18	46:50.3	3.4	2.8
22:37:04	46:42.3	3.0	2.7
22:37:57	46:33.9	2.1	2.5

demanding and simpler to program on hand-held calculators than the method of [1], and hence is somewhat more practical. It also has the advantage that it does not suffer from possible sensitivity to the initial position chosen to start the iterative process. It is stated in [1] that iterative methods are "likely to be used only by navigators with a strong mathematical interest"; however, we find them to be quite useful and have programmed the above equations on an HP-48SX calculator. We will make the code available to interested persons (send a self-addressed stamped envelope to: Thomas R. Metcalf, Institute for Astronomy, 2680 Woodlawn Dr., Honolulu, HI 96822). The program can be made quite flexible, and applies to situations much more general than that of a sequence of observations of a single body taken over a short period of time.

ACKNOWLEDGMENT

The authors wish to acknowledge the thorough and helpful comments of Dr. Robert W. Severance, especially in addressing the question of the robustness of the algorithm.

APPENDIX A

MATHEMATICAL BASIS OF THE ALGORITHM

Letting n denote the number of observations and defining the quantities u, v, and w as

$$u = \sin(L)$$
$$v = \cos(L)\cos(\lambda)$$
$$w = \cos(L)\sin(\lambda),$$

the least-squares problem can be stated as

$$\min_{u,v,w} \sum_{i=1}^{n} [\sin(d_i)u + \cos(d_i)\cos(GHA_i)v + \cos(d_i)\sin(GHA_i)w - \sin(H_{0_i})]^2 \tag{A1}$$

subject to the additional constraint that

$$u^2 + v^2 + w^2 = 1.$$

Here, H_{0_i} is the corrected altitude of the celestial body, d_i the body's declination, and GHA_i the GHA of the body, all for the i^{th} observation.

Using the method of Lagrange multipliers and the additional definitions

$$A_i = \sin(d_i),\ B_i = \cos(d_i)\cos(GHA_i),\ C_i = \cos(d_i)\sin(GHA_i),\ D_i = \sin(H_{0_i}),$$

the least-squares problem becomes

$$\min_{u,v,w,\mu} \left[\sum_{i=1}^{n} (A_iu + B_iv + C_iw - D_i)^2 - \mu(u^2 + v^2 + w^2 - 1) \right], \tag{A2}$$

where μ is the Lagrange multiplier, and the sum is over all the observations. Note that $A_i^2 + B_i^2 + C_i^2 = 1$.

Differentiating equation (A2) with respect to the four variables u, v, w, and μ, the linear equations that may be solved for u, v, and w have the matrix form

$$(\mathbf{G} - \mu\mathbf{I})\mathbf{x}(\mu) = \mathbf{r} \tag{A3}$$

where $\mathbf{I}$ is the 3×3 identity matrix,

$$\mathbf{G} = \begin{pmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C} \\ \mathbf{B}\cdot\mathbf{A} & \mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{C}\cdot\mathbf{A} & \mathbf{C}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{C} \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \mathbf{A}\cdot\mathbf{D} \\ \mathbf{B}\cdot\mathbf{D} \\ \mathbf{C}\cdot\mathbf{D} \end{pmatrix}, \quad \text{and } \mathbf{x}(\mu) = \begin{pmatrix} u(\mu) \\ v(\mu) \\ w(\mu) \end{pmatrix}$$

(the dependence of $\mathbf{x}$ on the parameter μ is emphasized here by writing $\mathbf{x}(\mu)$). The notation $\mathbf{A}$ indicates the n-vector $(A_1, \ldots, A_n)$, etc. Also, since each of the squares of A_i, B_i, and C_i add to one, the sum of the diagonal entries of $\mathbf{G}$ will equal n.

To determine μ, the nonlinear constraint equation

$$\|\mathbf{x}(\mu)\|^2 - 1 = 0 \tag{A4}$$

must be solved (here $\|\mathbf{x}\|$ denotes the length of $\mathbf{x}$, that is, $\sqrt{\mathbf{x}\cdot\mathbf{x}}$). By using the determinant solution to the system of equations (A3), it may be seen that this equation is equivalent to a sixth-degree polynomial in μ. Equations (A3) may be rewritten

$$\mu\mathbf{x} = \mathbf{G}\mathbf{x} - \mathbf{r},$$

where $\|\mathbf{x}\| = 1$. Since the residual difference $\|\mathbf{G}\mathbf{x} - \mathbf{r}\|$ is to be as small as possible, it is the *smallest root* (in absolute value) which we seek for μ. Thus, the least-squares problem is solved by the simultaneous solution of a linear system of equations for $u(\mu)$, $v(\mu)$, and $w(\mu)$, and a single nonlinear equation for μ.

In order to solve equations (A3) and (A4) efficiently, we will use the eigenvalues and eigenvectors of $\mathbf{G}$, since these are independent of the parameter μ and need be computed only once during any iteration process determining the value of μ. Denote by α_1, α_2, and α_3 the eigenvalues of $\mathbf{G}$, and by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ the corresponding eigenvectors. Since $\mathbf{G}$ is symmetric and positive-definite, the eigenvalues are real and satisfy

$$0 < \alpha_1 < \alpha_2 < \alpha_3.$$

They may be determined as the roots of the cubic equation $\det(G - \alpha I) = 0$, that is,

$$\begin{aligned} &-\alpha^3 + n\alpha^2 - [(G_{11}G_{22} - G_{12}^2) + (G_{11}G_{33} - G_{13}^2) + (G_{22}G_{33} - G_{23}^2)]\alpha \\ &+ [(G_{12}G_{23} - G_{13}G_{22})G_{13} - (G_{11}G_{23} - G_{12}G_{13})G_{23} + (G_{11}G_{22} - G_{12}^2)g_{33}] = 0. \end{aligned}$$

(For a good source on the solution of the cubic equation, see [6].) Also, because of the symmetry of $\mathbf{G}$, the eigenvectors will be orthogonal, that is, $\mathbf{e}_i \cdot \mathbf{e}_j = 0$

whenever $i \neq j$. These eigenvectors may be determined from subdeterminants of **G** as follows:

$$\mathbf{e}_i = \begin{pmatrix} \det\begin{pmatrix} G_{12} & G_{13} \\ G_{22} - \alpha_i & G_{23} \end{pmatrix} \\ -\det\begin{pmatrix} G_{11} - \alpha_i & G_{13} \\ G_{21} & G_{23} \end{pmatrix} \\ \det\begin{pmatrix} G_{11} - \alpha_i & G_{12} \\ G_{21} & G_{22} - \alpha_i \end{pmatrix} \end{pmatrix} \tag{A5}$$

for $i = 1, 2, 3$. Note that, in this form, the eigenvectors are not normalized and $\|\mathbf{e}_i\|$ does not in general equal one. Further, when all observations are of a single body over a short period of time, the smallest eigenvalue α_1 will be a very small positive number; this follows from the near singularity of the matrix **G** (the vectors **A**, **B**, and **C** are all nearly multiples of a vector of ones).

The vectors **x** and **r** may be expanded in terms of the eigenvectors of G:

$$\mathbf{x}(\mu) = \sum_{i=1}^{3} \gamma_i(\mu)\mathbf{e}_i \quad \text{and } \mathbf{r} = \sum_{i=1}^{3} \frac{\mathbf{r} \cdot \mathbf{e}_i}{\|\mathbf{e}_i\|^2}\mathbf{e}_i,$$

where $\gamma_i(\mu)$ ($i = 1, 2, 3$) are chosen so that equations (A3) are satisfied. In this way it may be determined that

$$\gamma_i = \frac{\mathbf{r} \cdot \mathbf{e}_i}{\|\mathbf{e}_i\|^2}\frac{1}{\alpha_i - \mu} \quad \text{for } i = 1, 2, 3.$$

Setting $\beta_i = (\mathbf{r} \cdot \mathbf{e}_i)/\|\mathbf{e}_i\|$, the equation for μ becomes

$$\left(\frac{\beta_1}{\alpha_1 - \mu}\right)^2 + \left(\frac{\beta_2}{\alpha_2 - \mu}\right)^2 + \left(\frac{\beta_3}{\alpha_3 - \mu}\right)^2 = 1, \tag{A6}$$

where α_i and β_i ($i = 1, 2, 3$) are constants independent of μ. In this form, it is clear that the equation for μ can be reduced to a sixth-degree polynomial. The general shape of the graph of this equation shows that the real roots will be distributed as follows:

$$1 \text{ root} < \alpha_1 < 2 \text{ (possible) roots} < \alpha_2 < 2 \text{ (possible) roots} < \alpha_3 < 1 \text{ root}.$$

Further, once μ has been determined, the solution for u, v, w is given by

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 + \gamma_3\mathbf{e}_3. \tag{A7}$$

The problem of solving (A6) may be attacked in several ways; Newton's method was found to be the most robust and can be used directly on (A6) or on the equivalent sixth-degree polynomial. These approaches were both tested, and the application of Newton's method directly to (A6) was deemed preferable in the context of this navigational problem. This method was indicated previously in the algorithm summary.

Finally, the position is computed from

$$L = \arcsin(u)$$

$$\lambda = \arctan\left(\frac{w}{v}\right). \qquad \text{(A8)}$$

Care must be taken in applying the arctangent function since most computers and calculators will return only the principal value of this function. If only the principal value is readily available, the ambiguity can be resolved as follows: if v is positive, the longitude is correct; however, if

$w > 0$ and $v < 0$, add 180 deg to λ

$w < 0$ and $v < 0$, subtract 180 deg from λ.

The above equations give north latitude and west longitude as positive angles, and south latitude and east longitude as negative angles.

APPENDIX B

ERROR ESTIMATE FOR OBSERVATIONS OF A SINGLE BODY

The error estimate for the position fix from observations of a single body is derived from a consideration of the length of time the observations span and the rate of change of the azimuth (deg/h):

$$\frac{dZ}{dt} = \frac{1}{\cos(H)}\sqrt{225\cos^2(d) - \left(\frac{dH}{dt}\right)^2}.$$

The position error in miles (ϵ) is found from a simple geometric analysis of the change in azimuth (δZ) between two lines of position derived from altitude observations with an error of δH arcmin,

$$\delta Z \approx \frac{57.3\,\delta H}{\epsilon}.$$

The factor 57.3 is a result of using $\sin(\delta Z) \approx \delta Z$ and converting to degrees. For n observations with Gaussian errors, the factor δH should be replaced by $\delta H/\sqrt{n-1}$. These results are combined to give equation (3).

REFERENCES

1. Severance, R. W., *Overdetermined Celestial Fix by Iteration*, NAVIGATION, Journal of The Institute of Navigation, Vol. 36, No. 4, Winter 1989, pp. 373–78.
2. Burch, D., *Celestial Navigation with the 2102-D Star Finder*, Sausalito, CA: Paradise Cay Publications, 1984.
3. Watkins, R. and Janiczek, P. M., *Sight Reduction with Matrices*, NAVIGATION, Journal of The Institute of Navigation, Vol. 25, No. 4, Winter 1978–79, pp. 447–48.
4. Moody, Alton B., *Celestial Navigation in the Computer Age*, New York: Van Nostrand Reinhold, 1982.

5. Doggett, L. E., Tangren, W. J., and Panossian, S. P., *Almanac for Computers, 1990,* Washington, D.C.: United States Naval Observatory, 1990.
6. Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., *Numerical Recipes in C, The Art of Scientific Computing,* Cambridge: Cambridge University Press, 1988.

NAVIGATION: *Journal of The Institute of Navigation*
Vol. 39, No. 4, Winter 1992–93
Printed in U.S.A.

The Institute's Professional Forum

An Extension to the Overdetermined Celestial Fix

THOMAS R. METCALF

Institute for Astronomy, University of Hawaii, Honolulu Hawaii

Received July 1992
Revised September 1992

ABSTRACT

I show that a previous algorithm that computes the overdetermined celestial fix by a least-squares fit includes nonequal weights for a set of observations of several celestial bodies, and I give an extension to the algorithm that weights the observations uniformly.

EXTENSION

In a previous paper [1], I presented a method for computing an overdetermined celestial fix. This method uses a least-squares fit to the sine of the observed altitude of one or more bodies as a function of time to compute the observer's position. In this paper, I show that the previous work includes a nonuniform weighting of the observations, and I give equations that weight the observations equally.

In [1], the least-squares fit is performed on the equation

$$\sin(H_c) = \sin(d)\sin(L) + \cos(d)\cos(L)\cos(GHA - \lambda), \tag{1}$$

where d is the observed body's declination, GHA is the body's Greenwich Hour Angle, L is the observer's latitude, and λ is the observer's longitude.

Since the least-squares fit is accomplished using the sine of the observed altitude, the algorithm minimizes the sum

$$\sum_{i=1}^{N} [\sin(H_{o_i}) - \sin(H_{c_i})]^2. \tag{2}$$

Here, H_{o_i} is the observed altitude for the i^{th} observation, and H_{c_i} is the computed altitude of the observed body at the position of the fix for the i^{th} observation. Since H_{o_i} is generally very close to H_{c_i}, and since the derivative of the sine function can be written

$$\frac{d\,\sin(h)}{dh} = \cos(h) = \lim_{h_1 \to h_2} \frac{\sin(h_1) - \sin(h_2)}{h_1 - h_2}, \tag{3}$$

we see that, using equation (3) in equation (2), the least-squares condition is equivalent to minimizing the sum

$$\sum_{i=1}^{N} [\sin(H_{o_i}) - \sin(H_{c_i})]^2 \approx \sum_{i=1}^{N} [H_{o_i} - H_{c_i}]^2 \cos^2(H_{o_i}). \qquad (4)$$

Thus, the least-squares fit in [1] effectively weights the observations by the square of the cosine of the observed altitude: the higher the observed altitude, the less the observation affects the solution. Figure 1 shows the weighting of the observations in the algorithm of [1] as a function of the observed altitude. The horizontal axis gives the observed altitude in degrees, and the vertical axis gives the effective weight of the observation in the least-squares fit. In a sense, this weighting is beneficial since most celestial sights become more difficult as the altitude increases, and the algorithm of [1] naturally weights the lower sights more than the higher ones. Note that for rapid observations of a single body, the observed altitudes, and hence the weights, are nearly uniform, and the extension presented below is relatively unimportant.

However, if the weighting is deemed undesirable by the navigator, for example, with sights of several bodies at widely varying altitudes, it is simple to modify the algorithm such that the sights are equally weighted. In effect, this entails using a least-squares fit not to the sine of the altitude, but to the altitude itself.

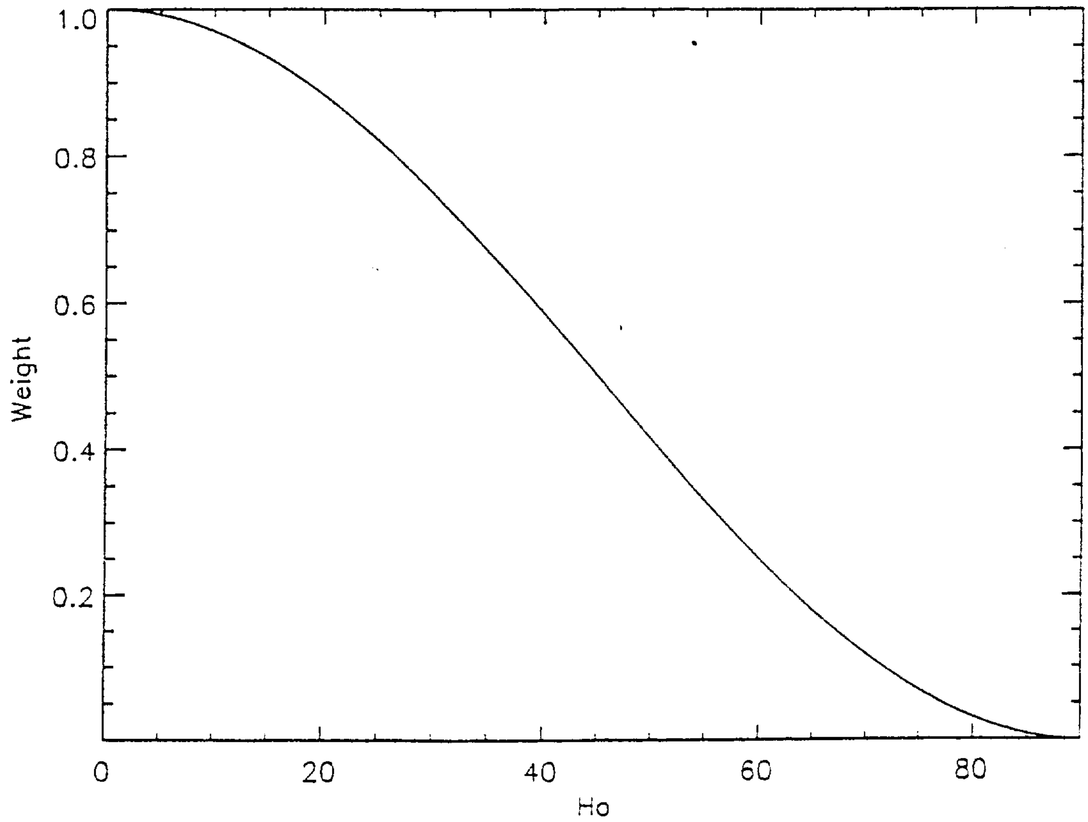

Fig. 1—Cos² Weighting of the Observations

Since the sights are weighted by the square of the cosine of the altitude in [1], simply including an additional weighting of $[1/\cos(H_{o_i})]^2$ in the fit will give the sights equal weight. To accomplish this, replace the "Basic Observational Sums" of [1] by

$$
\begin{aligned}
G_{11} &= \sum_{i=1}^{N} \omega_i \sin^2(d_i) & r_1 &= \sum_{i=1}^{n} \omega_i \sin(d_i)\sin(H_{o_i}) \\
G_{12} &= \sum_{i=1}^{N} \omega_i \sin(d_i)\cos(d_i)\cos(GHA_i) & r_2 &= \sum_{i=1}^{N} \omega_i \cos(d_i)\cos(GHA_i)\sin(H_{o_i}) \\
G_{13} &= \sum_{i=1}^{N} \omega_i \sin(d_i)\cos(d_i)\sin(GHA_i) & r_3 &= \sum_{i=1}^{N} \omega_i \cos(d_i)\sin(GHA_i)\sin(H_{o_i}) \\
G_{22} &= \sum_{i=1}^{N} \omega_i \cos^2(d_i)\cos^2(GHA_i) & & \\
G_{23} &= \sum_{i=1}^{N} \omega_i \cos^2(d_i)\sin(GHA_i)\cos(GHA_i) & G_{33} &= n - G_{11} - G_{22}
\end{aligned}
\tag{5}
$$

where

$$
n = \sum_{i=1}^{N} \omega_i
$$

$$
\omega_i = \left[\frac{1}{\cos(H_{o_i})}\right]^2 .
$$

Note that n represents the number of observations in [1], but has been redefined here. In this extension, n represents the sum of the weights, ω_i, while the new symbol N now represents the number of observations. This new definition of n should be maintained throughout the remaining equations in [1].

There is one problem with this extension: as an observed altitude approaches 90 deg, $\cos(H_{o_i})$ approaches zero, and ω_i becomes infinite. Hence, for observations above some limiting altitude, ω_i must be set to 1 rather than $[1/\cos(H_{o_i})]^2$. The limiting altitude is generally greater than 84 deg and is always less than 90 deg, the exact value depending on the numerical precision of the computer used. A useful empirical expression for the limiting altitude (H_{max}) is

$$
\cos^2(H_{max}) \approx 10^{9.7-p}, \tag{7}
$$

where p is the number of decimal digits of precision carried by the computer. For example, a computer that carries 12 decimal digits would have a limiting altitude of about 86 deg, while a computer that carries 15 decimal digits would have a limiting altitude of about 89.9 deg. Observations above this limiting altitude will still be weighted by the square of the cosine of the observed altitude since ω_i is set to 1.

CONCLUSION

This simple extension to the algorithm of [1] gives equal weight to the observations in the least-squares fit, a feature some navigators may find more appealing. Note that the ω_i defined above could take on any value; the extension here is only one example. If the navigator wishes to weight the solutions in some other way, any weighting scheme can be introduced into the ω_i.

REFERENCE

1. Metcalf, T. R. and Metcalf, F. T., *On the Overdetermined Celestial Fix*, NAVIGATION, Journal of The Institute of Navigation, Vol. 38, No. 1, Spring 1991, pp. 79–89.

NAVIGATION: *Journal of The Institute of Navigation*
Vol. 38, No. 3, Fall 1991
Printed in U.S.A.

Advancing Celestial Circles of Position

THOMAS R. METCALF
Institute for Astronomy, University of Hawaii, Hawaii
Received January 1991
Revised May 1991

ABSTRACT

This paper presents rigorous equations, useful with computer reduction of celestial sights, which correct the Greenwich Hour Angle and declination of a celestial body for the motion of a vessel. Advancing a circle of position in this way maintains the relationship between the geographical position of the body and the vessel, and hence is the best method for advancing an observation.

INTRODUCTION

When computing a running fix without the aid of a computer, one generally proceeds graphically by advancing a line of position (LOP) on a chart. To the extent that an LOP is a good approximation to the true circle of position, this technique works well. However, the graphical approach does not yield an efficient algorithm when calculating a running fix with a computer. Instead, to allow for the motion of the observer, the program adjusts the observations, not the LOP directly.

The simplest method of advancing an observation with a computer involves a correction to the observed altitude of a celestial body [1, 2]. However, this is clearly an approximation since it may alter the direction of the LOP, particularly for observations of bodies near the zenith. To advance the entire *circle* of position, and hence maintain the orientation of the LOP, one corrects the Greenwich Hour Angle (GHA) and declination of the observed body. By correcting the GHA and declination rather than the altitude, the circle of position is "dragged" along with the vessel, making the correction exact. An approximate technique for advancing the GHA and declination that is adequate for almost all situations encountered in navigation is presented in [1]. However, a rigorous formulation is not much more demanding computationally and, with the advent of the handheld computer, may be of interest.

Table 1 indicates typical errors incurred when using the approximate techniques discussed above to advance the observation of a body near the zenith (85°17′). The first column gives the distance traveled from the starting point (45° N, 30.3° W). The second column shows the error in the geographical position (GP) of the fictitious body when the GHA and declination are advanced using the approximate formulae given in [1]. The error in a derived position could be substantially larger for LOPs that cross at small angles, as in the case of multiple sights of a single body over a relatively short period of time [3, 4]. The

Table 1—Example: Errors Incurred in Approximate Techniques

(Position: 45° N, 38°20′ W; Course: 45° (NE); GHA/declination: 45°, 45° N)

Distance Traveled (nmi)	Error in GP Computed from [1] (nmi)	Azimuth Error from Altitude Correction (deg)
60	0.06	−7.1
120	0.53	−12.3
180	1.44	−16.3
300	4.60	−21.6
480	12.98	−25.8
600	21.20	−27.2

third column shows the error in the orientation of an LOP computed by altering the observed altitude of the body to correct for observer motion. Again, the error in the derived position could be considerable when the LOPs intersect at small angles.

METHOD

Since the geographical position of the observed body is not necessarily near the vessel, the correction cannot be made by simply moving the GP the same distance along the same course as the vessel's motion. Instead, the coordinate system defining GHA and declination is mathematically rotated by an amount equal to the vessel's change in latitude and longitude, since any rotation will preserve the relative position of the GP and the vessel. In other words, rather than treating the vessel as moving over a fixed surface, the vessel and the GP are considered fixed, with the earth moving underneath them in such a way that the vessel begins and ends at the appropriate places. The final position of the GP after this rotation is the advanced GP we seek.

The usual convention is to describe the GP with two angles: the GHA and the declination. However, the rotation takes a simpler form if the GP is represented as a vector directed from the center of the earth to the actual GP on the surface. With the axis of the rotation represented vectorally by $\hat{\mathbf{n}}$ (the "^" indicates a unit vector) and the angle of the rotation by θ, the vector coordinates of the GP after rotation ($\hat{\mathbf{r}}'$) are related to the vector coordinates before rotation ($\hat{\mathbf{r}}$) by

$$\hat{\mathbf{r}}' = \hat{\mathbf{r}}\cos(\theta) + \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})[1 - \cos(\theta)] + (\hat{\mathbf{r}}\times\hat{\mathbf{n}})\sin(\theta) \qquad (1)$$

where "·" is the vector dot-product and "×" is the vector cross-product [5]. Since the vessel's position changes in both latitude and longitude, the simplest method of rotating the coordinate system is to rotate the vessel's position first in longitude and then in latitude. The longitude rotation is trivial as it is simply a rotation about the earth's pole, and equation (1) need only be used for the latitude rotation.

The rotation angle θ is the change in longitude or latitude the vessel undergoes. For a rhumb line (neglecting the ellipticity of the earth) [6],

$$\theta_{\text{long}} = -\sin(\alpha)\int_0^D \frac{1}{\cos\,(L + x\cos(\alpha))}\,dx$$
$$= -\frac{180}{\pi}\tan(\alpha)\left[\ln\left(\frac{\tan\left(45^\circ + \frac{L+D\cos(\alpha)}{2}\right)}{\tan\left(45^\circ + \frac{L}{2}\right)}\right)\right] \qquad (2)$$

and

$$\theta_{\text{lat}} = D\cos(\alpha)\,. \qquad (3)$$

Here, L represents the initial latitude of the vessel in degrees, α represents the true course steered in degrees clockwise from north, and D represents the distance traveled in nautical miles divided by 60 (i.e., in degrees). Note that the equation for θ_{long} becomes indeterminate when $|\cos(\alpha)|$ is small. In that case, θ_{long} can be approximated by

$$\theta_{\text{long}} = \frac{-D\sin(\alpha)}{\cos(L)} \qquad (4)$$

when $|D\cos(\alpha)| \ll \frac{180}{\pi}|\cot(L)|$. I have assumed positive west longitude and north latitude here and throughout the paper.

Using a three-dimensional Cartesian coordinate system with the first two axes in the plane of the earth's equator and the third axis along the earth's rotational axis, $\hat{\mathbf{n}}$ for the latitude rotation is given by

$$\hat{\mathbf{n}} = \begin{bmatrix} \cos(90^\circ + \lambda + \theta_{\text{long}}) \\ \sin(90^\circ + \lambda + \theta_{\text{long}}) \\ 0 \end{bmatrix}, \qquad (5)$$

where λ is the initial longitude of the observer, and the trivial longitude rotation has already been included. The unit coordinate vector of the GP is given by

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos(d)\cos(GHA + \theta_{\text{long}}) \\ \cos(d)\sin(GHA + \theta_{\text{long}}) \\ \sin(d) \end{bmatrix}, \qquad (6)$$

where d is the initial declination of the body, and GHA is the initial Greenwich Hour Angle. Since equations (5) and (6) already account for the longitude rotation, application of equation (1), with $\theta = \theta_{\text{lat}}$, yields the final values of the GHA and declination.

To extract the new GHA (GHA′) and declination (d′) after rotation, use

$$d' = \arcsin(r_3')$$
$$GHA' = \arctan\left(\frac{r_2'}{r_1'}\right). \qquad (7)$$

Here the subscripts 1, 2, 3 refer to the first, second, and third elements of $\hat{\mathbf{r}}'$, respectively. Care must be taken with the arctan function since most computers will return only the principal value, but the GHA can take on any value from 0 to 360 deg.

The vector formulation in equation (1) is particularly useful for computers that incorporate vector mathematics. If such a simplification is not available, the quantities in equation (7) can be reduced to

$$\begin{aligned} r_3' &= \sin(d)\cos(\theta_{lat}) + \cos(d)\sin(\theta_{lat})\cos(LHA) \\ r_2' &= \cos(\theta_{lat})\sin(GHA + \theta_{long}) + \cos(\lambda + \theta_{long})\sin(LHA)\,(1 - \cos(\theta_{lat})) \\ &\quad - \sin(\lambda + \theta_{long})\tan(d)\sin(\theta_{lat}) \\ r_1' &= \cos(\theta_{lat})\cos(GHA + \theta_{long}) - \sin(\lambda + \theta_{long})\sin(LHA)\,(1 - \cos(\theta_{lat})) \\ &\quad - \cos(\lambda + \theta_{long})\tan(d)\sin(\theta_{lat}) \end{aligned} \tag{8}$$

where LHA is the local hour angle (GHA − λ). The approximation indicated in [1] follows from equation (8) when θ_{lat} is a small angle.

REFERENCES

1. Moody, A.B., *Celestial Navigation in the Computer Age,* New York: Van Nostrand Reinhold, 1982.
2. Doggett, L. E., Tangren, W. J., and Panossian, S. P., *Almanac for Computers, 1990,* Washington: United States Naval Observatory, 1990.
3. Metcalf, T. R. and Metcalf, F. T., *On the Overdetermined Celestial Fix,* NAVIGATION, Journal of The Institute of Navigation, Vol. 38, No. 1, Spring 1991, pp. 79–89.
4. Severance, R., *Overdetermined Celestial Fix by Iteration,* NAVIGATION, Journal of The Institute of Navigation, Vol. 36, No. 4, Winter 1989–90, pp. 373–78.
5. Goldstein, H., *Classical Mechanics,* Reading, MA: Addison-Wesley, 1981.
6. Bacon, R. H., *The Rhumb Line on the Ellipsoidal Earth,* NAVIGATION, Journal of The Institute of Navigation, Vol. 6, No. 7, Autumn 1959, pp. 470–74.

NAVIGATION: *Journal of The Institute of Navigation*
Vol. 41, No. 2, Summer 1994
Printed in U.S.A.

Piloting with Celestial Algorithms

THOMAS R. METCALF
Institute for Astronomy, University of Hawaii, Honolulu, Hawaii

FREDERIC T. METCALF
University of California, Riverside, California

Received February 1993
Revised October 1993

ABSTRACT

This paper shows how coastal piloting observations can be cast in a form that allows them to be reduced using a celestial navigation algorithm. This is useful when navigating with a computer that can compute a position fix from celestial observations, but is not explicitly designed to handle coastal piloting. It is also a useful technique for incorporating piloting into a computer program without programming piloting and celestial techniques separately. We treat three cases: a range to a known object, a bearing to a known object, and a horizontal angle measured between two known objects.

INTRODUCTION

This is the fourth paper in a series describing navigational techniques that are suitable for use with the now common hand-held computer [1–3]. In this paper, we discuss how celestial algorithms can be used in coastal piloting. With this technique there is no need to code separate algorithms for celestial and coastal position fixes, and the task of programming a complete navigation system is greatly simplified.

Most computers that solve celestial navigation problems fix an observer's position by solving for the intersection between two or more celestial circles of position [1, 3]. These circles are specified by the Greenwich hour angle (GHA), declination, and altitude (H_0) of the observed bodies. Several of the most common methods of piloting also involve circles of position: range to a known object, bearing to a known object, and the horizontal angle between two known objects. Since these observations yield circles of position, just like the celestial observations, the same algorithm used to solve for a celestial fix can be used to solve for a coastal fix, or for a combination of both. In fact, once the coastal observation has been input, it can be treated exactly as a celestial observation would be: it can be advanced, retarded, and combined with any other data to yield a fix or running fix.

To use these piloting observations in a celestial algorithm, we must first specify the circle of position in the same format as the celestial observation, i.e., as an equivalent GHA, declination, and H_0. Since such observations are *equivalent* celestial observations, in the sense that they yield the same circle of position as the original coastal observation, it is not necessary (and not appropriate) to correct the altitude in any way for such effects as dip and refraction. This is the case for all the equivalent observations in this paper. The GHA and declination specify the longitude and latitude of the center of the circle of position, and the altitude specifies its radius. In this paper, we show how the equivalent celestial observation is obtained for each of the three piloting observations mentioned above.

RANGE

In piloting, it is quite common to use the observed distance to a known object to compute a circle of position. Generally, the distance is computed from an observation of the vertical extent of the object [4], or is determined with radar. The circle of position is centered on the observed object, and the radius of the circle is equal to the observed distance. This makes the computation of the equivalent celestial observation very simple. If the position of the observed object is (λ, l), where λ is the longitude (west is positive here and throughout this paper), and l is the latitude (north is positive here and throughout the paper), and if the distance to the object in nautical miles is d, then

$$\begin{aligned} \text{GHA} &= \lambda \text{ (add } 360^\circ \text{ if } \lambda < 0) \\ \text{declination} &= l \\ H_0 &= 90^\circ - \frac{d}{60} \end{aligned} \tag{1}$$

These values for the GHA, declination, and H_0 simulate a celestial observation with the same circle of position as the range observation.

BEARING

We next examine the case in which the bearing to a known object is observed. Since visible light and radio waves travel along great circles, the equivalent celestial observation must yield a circle of position that is a great circle: the great circle passing through both the known object and the observer's position. A celestial observation with altitude 0° yields a great circle of position. Hence, the altitude of the equivalent celestial observation must be zero. The GHA and declination of the equivalent celestial observation, which give the center of the circle of position, are the coordinates of a pole of this great circle.

Since the observer's position is unknown, this description of the great circle is inadequate. However, if the known object and the observer are close, then an equivalent description of this great circle would be to view it as the great circle that passes through the known object with the same bearing as the observed bearing. This does not require knowledge of even a dead reckoning position. If the known object and the observer are not close (e.g., as in the case of a radio bearing), then the bearing at the known object does not match that at the observer, since bearings generally change along great circles. In this case, a correction must be applied to determine the correct bearing at the known object, and this can be accomplished through knowledge of a dead-reckoning position. The *correction* will be taken to be the initial great circle course from the object to the dead-reckoning position, minus the initial great circle course from the dead-reckoning position to the object. This will convert the bearing from the observer to the object into a bearing from the object to the observer.

We represent the location of the observed object by $P_{obs} = (\lambda_{obs}, l_{obs})$, the observed *true* bearing from the observer to the object by C_{obs} $(0° \leq C_{obs} < 360°)$, and the observer's dead reckoning position by $P_{dr} = (\lambda_{dr}, l_{dr})$.

To compute the correction to the observed bearing, we need to compute the initial great circle course from the dead-reckoning position to the observed object and vice versa. Denoting the initial course along the great circle from an arbitrary location $P_1 = (\lambda_1, l_1)$ to a second arbitrary location $P_2 = (\lambda_2, l_2)$ by $C_{gc}(P_1, P_2)$, this initial great circle course is given by [4]:

$$C_{gc}(P_1, P_2) = \arctan\left(\frac{\sin(\lambda_1 - \lambda_2)}{\cos(l_1)\tan(l_2) - \sin(l_1)\cos(\lambda_1 - \lambda_2)}\right) \tag{2}$$

Care must be taken with this formula to ensure that C_{gc} is in the correct quadrant: if the denominator is negative, then 180° must be subtracted when the numerator is negative and added when the numerator is positive.

Using this expression for the initial great circle course, the corrected bearing is

$$C = C_{obs} + C_{gc}(P_{obs}, P_{dr}) - C_{gc}(P_{dr}, P_{obs}). \tag{3}$$

If $C < 0°$, add 360° to C, and if $C \geq 360°$, subtract 360° from C, so that $0° \leq C < 360°$.

With the corrected bearing, the equivalent celestial observation is

$$\begin{aligned} &\text{GHA} = \lambda_v \text{ (add } 360° \text{ if } \lambda_v < 0) \\ &\text{declination} = l_v \pm 90° \; (+ \text{ if } l_v \leq 0 \text{ and } - \text{ if } l_v > 0) \\ &H_o = 0 \end{aligned} \tag{4}$$

where (λ_v, l_v) is one of the two vertices of the great circle (points of extreme latitude). A vertex of the great circle can be computed from [4]:

$$\begin{aligned} \lambda_v &= \lambda_{obs} \pm \arcsin\left(\frac{\cos(C)}{\sin(l_v)}\right) \\ l_v &= \pm \arccos\left(|\cos(l_{obs})\sin(C)|\right) \end{aligned} \tag{5}$$

The "±" in the equation for λ_v is positive if C is greater than 180° and negative if C is less than 180°. The "±" in the equation for l_v is positive if l_{obs} is positive (the observed object is in the northern hemisphere) and negative if l_{obs} is negative (the observed object is in the southern hemisphere).

HORIZONTAL ANGLE

The third piloting observation that yields a circle of position is an observation of the horizontal angle between two known objects [4]. In fact, the circle of position for the horizontal angle observation is a partial circle, not a complete one. Since the equivalent celestial observation will yield an entire circle, some of the equivalent circle is invalid. However, this should not be a problem since the circle of position must be crossed with another piloting or celestial circle of position to compute a position fix.

When the horizontal angle between two known objects is observed, the line of position consists of two circular arcs passing through both observed objects. The centers of the circles lie along the perpendicular bisector of the line connecting the two objects (see Figure 1). Simply by knowing on which side of the objects the observer's position is located, one of the arcs can be eliminated, and the line of position is then a single circular arc. Figure 1 shows the two important cases where the measured horizontal angle is acute (less than 90°) and obtuse (greater than 90°). In the case of the acute angle, the correct arc is the one with the center closest to the dead-reckoning position. For the obtuse angle, the arc with its center farthest from the dead-reckoning position is correct.

Representing the distance in nautical miles between the two observed objects by 2a and the horizontal angle observed between the two objects by A, the radius of the circular arc is $a / \sin A$; the center of the arc is located $\pm\, a \cot A$ from the line connecting the two objects and is on the perpendicular bisector of that line. Note that a is half the distance between the two observed objects.

Denoting the bearing from Object 1 to Object 2 by C_{12}, a computationally simpler method of locating the two centers is to notice that the directions from Object 1 to the centers of the circles are $C_{12} \pm (90^\circ - A)$. We then use the technique of [2] to translate the position of Object 1 a distance equal to the radius of the circle along these directions to find the centers.

The coordinates of the center of the circle give the GHA and declination of the equivalent celestial observation, and the radius gives the altitude. With the position of the two observed objects given by (λ_1, l_1) and (λ_2, l_2), the equivalent celestial observation is

$$\text{GHA} = \lambda_1 - \frac{a}{60 \sin A} \, \frac{\sin (C_{12} \pm (90^\circ - A))}{\cos (l_1)}$$

$$\text{declination} = l_1 + \frac{a}{60 \sin A} \cos (C_{12} \pm (90^\circ - A)) \, .$$

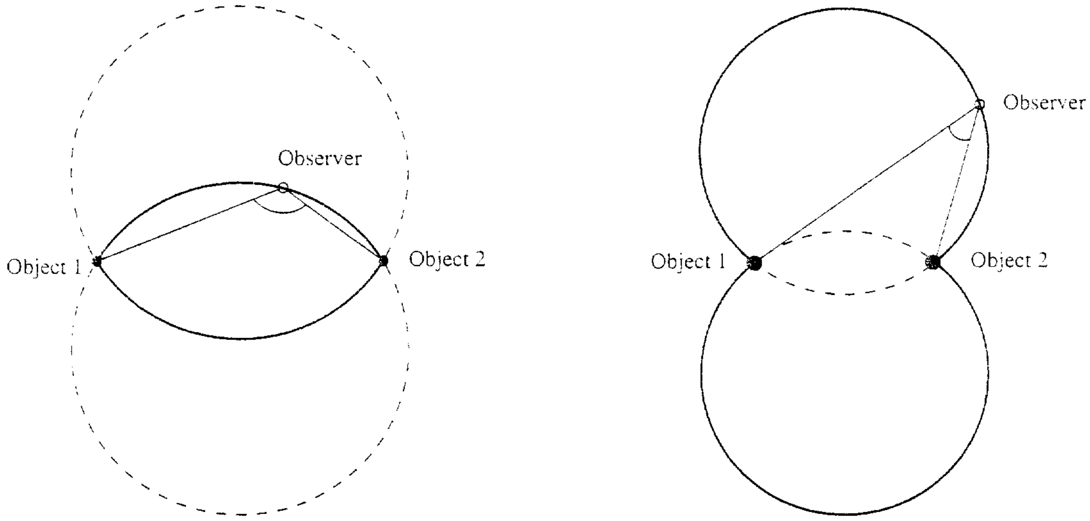

Fig. 1—The Circles of Equal Angle

$$H_o = 90° - \frac{a}{60 \sin A} \tag{6}$$

For a rhumb line, C_{12} can be computed from

$$C_{12} = \arctan\left(\frac{\frac{\pi}{180}(\lambda_1 - \lambda_2)}{\ln\left(\frac{\tan(45° + l_2/2)}{\tan(45° + l_1/2)}\right)}\right), \tag{7}$$

while a can be calculated from

$$a = \begin{cases} \frac{1}{2}\left|\frac{l_2 - l_1}{\cos(C_{12})}\right| 60, & \text{if } |\cos(C_{12})| > 0.0001 \text{ (say)}, \\ \frac{1}{2}\left|\frac{(\lambda_2 - \lambda_1)\cos((l_1 + l_2)/2)}{\sin(C_{12})}\right| 60, & \text{otherwise.} \end{cases} \tag{8}$$

As with equation (2), care must be taken to ensure that C_{12} is in the correct quadrant. Care must also be exercised to ensure that $\lambda_1 - \lambda_2$, in equation (7), is in the range $-180° \le \lambda_1 - \lambda_2 \le 180°$, by adding or subtracting 360° from $\lambda_1 - \lambda_2$. Similarly, $\lambda_2 - \lambda_1$ must be in the range $-180° \le \lambda_2 - \lambda_1 \le 180°$ in equation (8). Note that equation (6) assumes that the observer is not too far from the observed objects. See the appendix for the criteria that must be met to make equation (6) valid.

Because of the "±" operators, there are two solutions for the GHA and declination in equation (6). These two solutions are for the two arcs discussed above. The solution closest to the dead-reckoning position should be selected if the observed horizontal angle is acute, and the solution farthest from the dead-reckoning position should be selected if the observed horizontal angle is obtuse. See the appendix for the derivation of the equivalent celestial observation.

EXAMPLE

Suppose that we obtain the following observations: a range, a bearing, and a horizontal sextant angle. What are the equivalent celestial observations if the dead reckoning position is (117°41' W, 33°27' N)?

Observation 1:
Object: East End of Santa Catalina Island (118°20.0' W, 33°18.5' N)
Observation: Range = 31.6 nmi

Observation 2:
Object: Santiago Peak (117°31.9' W, 33°42.5' N)
Observation: Bearing = 28.5° true

Observation 3:
Objects: San Onofre (117°33.5' W, 33°22.5 N) and Santiago Peak
Observation: Horizontal angle = 102°

Table 1 shows the equivalent celestial observations derived using the techniques in this paper. The position computed from these observations is (117°41.8' W, 33°26.4' N).

APPENDIX
THE CIRCLE OF EQUAL ANGLE

Suppose two distinct objects are located at (−a, 0) and (a, 0) in a planar coordinate system. If the angle subtended by these objects at the position of an observer is A, then the observer must be located at a point (x, y) satisfying the law of cosines (Figure 2):

$$(2a)^2 = \left[\sqrt{(x + a)^2 + y^2}\right]^2 + \left[\sqrt{(x - a)^2 + y^2}\right]^2$$
$$-2\sqrt{(x + a)^2 + y^2}\,\sqrt{(x - a)^2 + y^2}\cos A. \qquad \text{(A-1)}$$

We assume here that the distances between the observed objects and the observer are sufficiently small so that we can use plane geometry, say, less than 100 nmi or so. For distances less than 100 nmi, the error in the line of position is less than 50 m. The error in the derived position depends on the relative orientation of all lines of position used in the fix, exactly as in celestial observations [4].

The above precludes the use of distant radio bearings in this technique, but visual observations should not present any problems. Further manipulation gives

$$4a^2y^2\cos^2 A = (x^2 + y^2 - a^2)^2 \sin^2 A$$

Finally, since $A \neq 0$,

Table 1—Equivalent Celestial Observations

Observation	GHA	Declination	H_o
1	118°20.0′	33°18.5′ N	89°28.4′
2	224°21.3′	23°27.2′ N	0°00.0′
3	117°30.2′	33°32.3′ N	89°49.8′

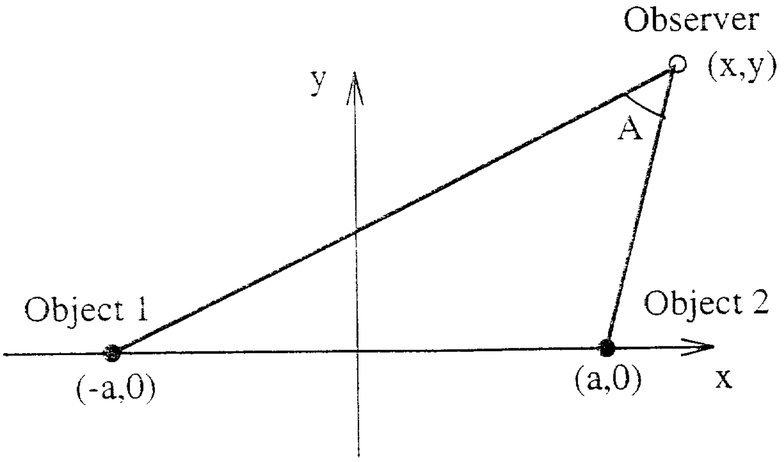

Fig. 2—The Geometry for the Derivation of the Circle of Equal Angle

$$x^2 + y^2 \pm 2a(\cot A)y - a^2 = 0 \quad \text{or}$$

$$x^2 + (y \pm a \cot A)^2 = (a/\sin A)^2 \tag{A-2}$$

the equation of a circle with radius $a / \sin A$ and center at $(0, \pm a \cot A)$.

Note that the points $(\pm a, 0)$ both lie on each of the circles; i.e., the possible circles of position both pass through the locations of the two observed objects. The two cases corresponding to obtuse and acute observation angles are illustrated in Figure 1.

To find the centers of the two circles, denote the bearing from Object 1 to Object 2 by C_{12}. This bearing is computed from equation (2) or (7). Then, by simple geometry, the bearing from Object 1 to the centers of the circles, denoted by α, is

$$\alpha = C_{12} \pm (90^\circ - A) \tag{A-3}$$

Since Object 1 lies on the circle, the distance from the observed object to the center of the circle of position is simply the radius of the circle. Hence, we can find the centers by moving from Object 1 a distance equal to the radius of the circle in a direction given by α.

We assume that the radius is small enough such that

$$\frac{a}{60 \sin A}\cos(\alpha) \ll \frac{180}{\pi}|\cot(l_1)| \tag{A-4}$$

When this condition holds, the change in longitude is greatly simplified [2]. Since the above derivation of the circles of position assumed plane geometry, condition (A-4) will normally be satisfied when the distances involved are small enough for the technique to work. Hence, using the simplified equations for the change in longitude does not introduce a new restriction. However, near the poles where $\cot(l_1)$ becomes small, the translation from Object 1 to the centers of the circles should be done along a great circle, rather than the rhumb line used below.

Denoting the radius of the circles (in degrees) by $r = a/(60 \sin A)$, the change in longitude and latitude in moving from Object 1 to the centers of the circles is

$$\theta_{\text{long}} = -r\frac{\sin(\alpha)}{\cos(l_1)}$$

$$\theta_{\text{lat}} = r\cos(\alpha) \tag{A-5}$$

and the centers of the circles are

$$\left(\lambda_1 - r\frac{\sin(C_{12} \pm (90^\circ - A))}{\cos(l_1)},\ l_1 + r\cos(C_{12} \pm (90^\circ - A))\right) \tag{A-6}$$

The centers of the circles give the GHA and declination of the equivalent celestial observation. If A is acute, the center closest to the dead-reckoning position should be used. If A is obtuse, the center farthest from the dead-reckoning should be used (Figure 1).

REFERENCES

1. Metcalf, T. R. and Metcalf, F. T., *On the Overdetermined Celestial Fix,* NAVIGATION, Journal of The Institute of Navigation, Vol. 38, No. 1, Spring 1991, pp. 79–89.

2. Metcalf, T. R., *Advancing Celestial Circles of Position*, NAVIGATION, Journal of The Institute of Navigation, Vol. 38, No. 3, Fall 1991, pp. 285–88.
3. Metcalf, T. R., *An Extension to the Overdetermined Celestial Fix*, NAVIGATION, Journal of The Institute of Navigation, Vol. 39, No. 4, Winter 1992–93, pp. 477–480.
4. Bowditch, *The American Practical Navigator*, Vols. 1 and 2, Defense Mapping Agency/Topographic Center, Washington, D.C., 1984.