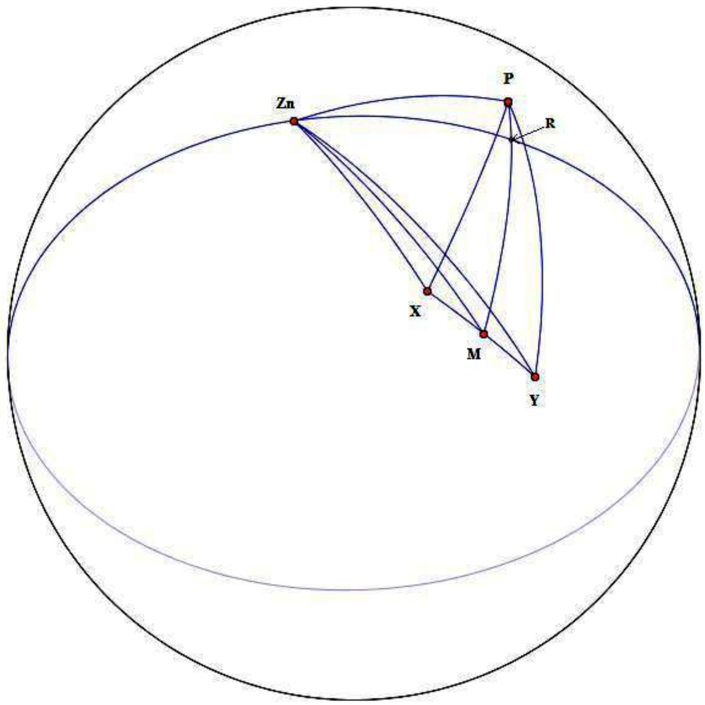
Derivation of Bowditch's method of double altitudes for the determination of the latitude

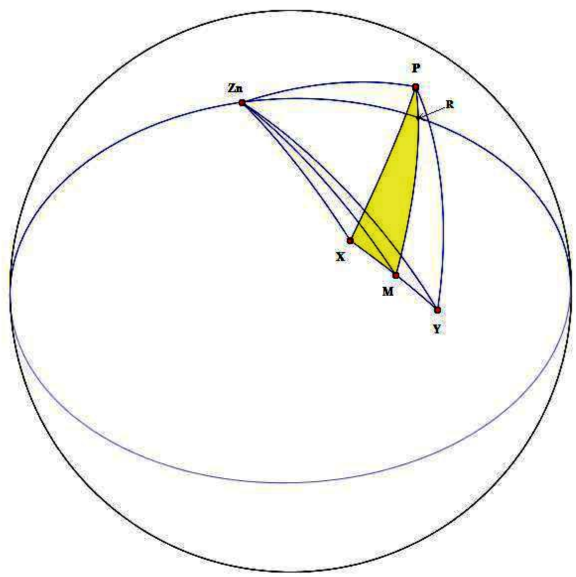
This note is an illustrated exposition of Benjamin Peirce's explanation of the derivation of Bowditch's method. It was published in Peirce's *Elementary Treatise on Plane and Spherical Trigonometry*, published in 1845. I have not yet traced down when the this method first appeared in Bowditch, but it was already included by 1825. The basic method is as follows: determine the corrected altitude of the sun a two points in time, noting the time elapsed between the first and second observations. You do not need to know the actual time of either observation, only the time that has passed between the two observations, as might be determined by a simple watch. The only other piece of information used is the sun's declination on that date. A series of logarithmic calculations lead to the latitude of the observer. In its simplest form, one assumes the observer has not moved between the two observations, but Bowditch describes the simple modifications needed to take into account movement of the ship between the two observations, using a dead reckonning of the change in the ship's position.

The basic construction is to connect the geographical positions of the two solar observations (X and Y) with the arc of the great circle that passes through the two points. Then connect the celestial north pole (P) with X and with Y; connect the zenith (Zn) with X and with Y; connect P with Zn.

Then construct two auxilliary lines, each of which serve the purpose of dividing an oblique spherical triangle into two right spherical triangles: the first of these lines bisects the angle XPY and extends to the arc of the great circle connecting X and Y. Lable the point of intersection M, since this line segment will be perpendicular bisector of XY. Lastly, drop a perpendicular from the zenith Zn to the line passing through P and M. Label the point at which this perpendicular meets PM as point R.



Peirce uses three right triangles (PMX, ZnRM, and ZnRP) and two oblique triangles (ZnMX and ZnMY) to derive his methods, using Napier's rules for the right triangles and the spherical law of cosines for the oblique triangles. In fact, of Napier's ten rules for right triangles, Peirce uses only two rules to determine the latitude. The first rule is equivalent to the law of sines for spherical triangles: the sine of the leg is to the sine of the opposite angle as the sine of the hypotenuse is to 1. The second rule he uses is that the cosine of the hypotenuse is equal to the product of the cosines of the legs.



Let us call the sun's declination D and the elapsed time expressed in angular measure t. Let h_1 be the altitide of the sun at position X, and let h_2 be the altitude of the sun at position Y .

Using the right triangle PMX (right angle at M), we note that hypotenuse $PX = 90° - D$ and the angle $XPM = t/2$. We introduce two new variables at this point. A is the length of leg XM and B is the declination (or latitude) of point M. Thus the two legs of triangle PMX are of length A and $90° - B$, while the hypotenuse, PX, is of length $90° - D$.

Napier's rules provide two equations:

$$\frac{\sin A}{\sin(t/2)} = \frac{\sin(90° - D)}{1}$$

and

$$\cos(90° - D) = \cos A \cos(90 - B)$$

or equivalently

$$\sin A = \cos D \sin \frac{t}{2}$$
$$\sin B = \sin D \sec A$$

.

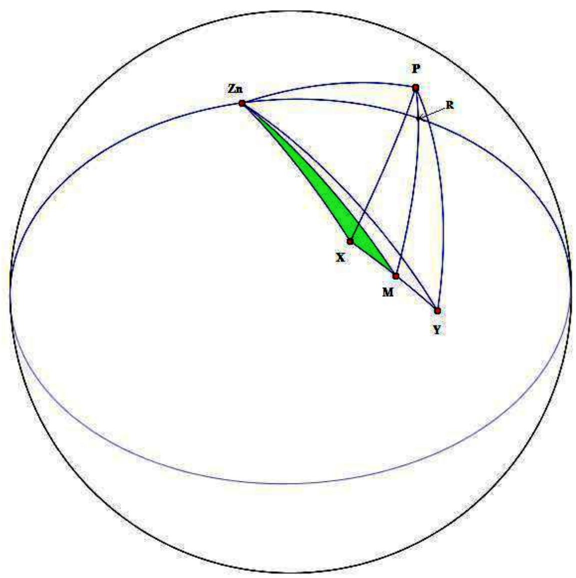
For reasons I do not understand, perhaps in order to avoid flipping pages back and forth in the log tables when future calculations are performed, Peirce and

Bowditch prefer to invert each equation:

$$\csc A = \sec D \csc \frac{t}{2} \tag{1}$$

$$\csc B = \csc D \cos A \tag{2}$$

We now turn our attention to the two oblique triangles, and in each case we apply the law of cosines. We introduce two auxilliary variables: θ to denote the angle ZnMP and F to denote the length of side MZn.

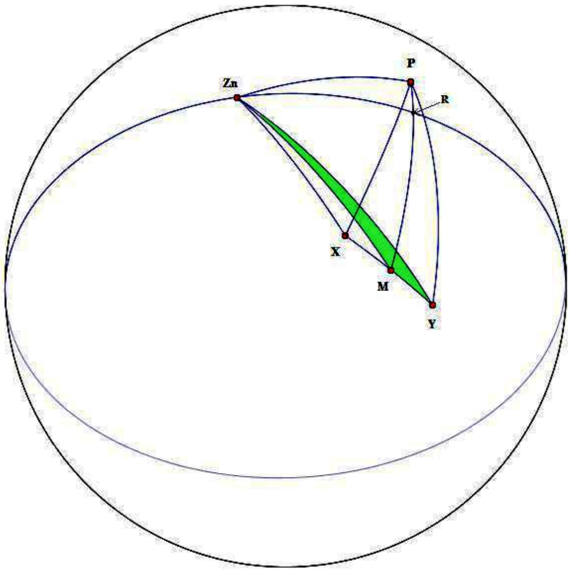


Side $XM = A$, side $XZn = 90° - h_1$, side $MZn = F$. Since angle $XMZn = 90°$ and θ denotes angle ZnMP, angle $XMZn = 90° - \theta$. Thus the law of cosines gives us

$$\cos(90° - h_1) = \cos A \cos F + \sin A \sin F \cos(90° - \theta)$$

or

$$\sin h_1 = \cos A \cos F + \sin A \sin F \sin \theta$$



Considering triangle $ZnMY$, the situation is similar. Side $ZnM = F$, side $MY = A$, and side $ZnY = 90° - h_2$. Recalling that θ denotes angle $ZnMP$, we note that angle $ZnMY$ is greater than 90° by amount θ whereas angle $ZnMX$ was less than 90° by θ. Therefore:

$$\cos(90° - h_2) = \cos A \cos F + \sin A \sin F \cos(90° + \theta).$$

Since $\cos(90 + x) = -\sin(x)$, this simplifies to:

$$\sin h_2 = \cos A \cos F - \sin A \sin F \sin \theta.$$

Adding the equation for $\sin h_1$ to the equation for $\sin h_2$ we get

$$\sin h_1 + \sin h_2 = 2 \cos A \cos F$$

Subtracting the equation for $\sin h_2$ from the equation for $\sin h_1$ we get

$$\sin h_1 - \sin h_2 = 2 \sin F \sin A \sin \theta$$

These equations can be rewritten in terms of the half sum and half difference of the two altitudes. Let $hs = \frac{h_1+h_2}{2}$ and $hd = \frac{h_1-h_2}{2}$.

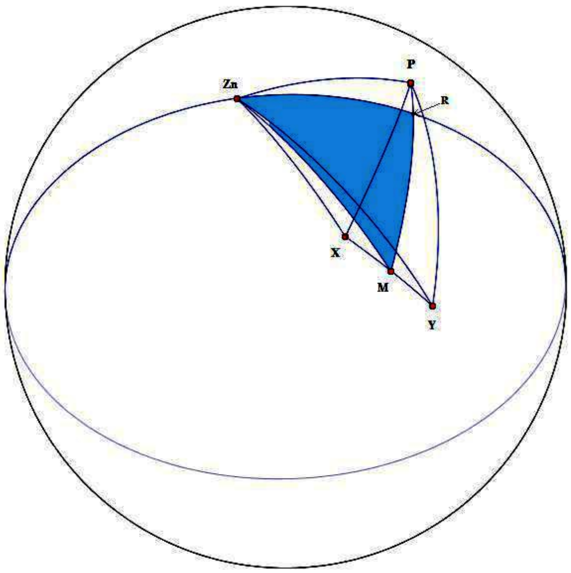
Then $\sin h_1 + \sin h_2 = 2 \sin hs \cos hd$ and $\sin h_1 - \sin h_2 = 2 \cos hs \sin hd$.

Using these subsitutions we get

$$\sin hs \cos hd = \cos A \cos F$$

$$\cos hs \sin hd = \sin A \sin F \sin \theta$$

Now we consider the right spherical triangle ZnMR (with right angle at M). The length of the hypotenuse , ZnM, has already been denoted by the variable F in the previous section. The angle at M has also previously been represented by the variable θ. We introduce variable C to represent the length of side ZnR and variable ζ to represent the length of side MR.



Then Napier's rules give

$$\cos F = \cos\zeta\cos C$$

$$\sin C = \sin F\sin\theta.$$

Substituting $\sin C$ for $\sin F\sin\theta$ in the equation $\sin C = \sin F\sin\theta$ we get

$$\cos hs\sin hd = \sin A\sin C$$

or

$$\sin C = \cos hs\sin hd\csc A. \tag{3}$$

Substituting $\cos C\cos\zeta$ for $\cos F$ in the equation $\cos F = \cos\zeta\cos C$ we get,

$$\sin hs\cos hd = \cos A\cos\zeta\cos C$$

or

$$\cos\zeta = \sin hs\cos hd\sec A\sec C$$

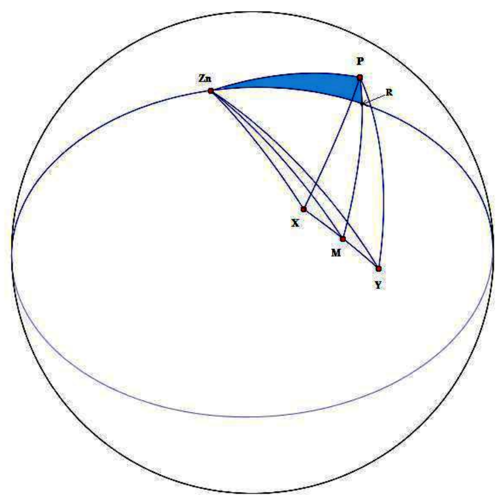
which inverted yields

$$\sec\zeta = \csc hs\sec hd\cos A\cos C. \tag{4}$$

We introduce another variable, E, to denote the declination of point R. Recalling that variable B denotes the declination of M and ζ the length of segment MR, it follows that since MR + RP = MP, $\zeta + (90° - E) = (90° - B)$. Solving for E, we get

$$E = B + \zeta \tag{5}$$

Finally we consider the right triangle ZnPR (with right angle at R). Hypotenuse ZnP is the complement of the desired latitude, side ZnR has the value C, and side RP has value $90° - E$.



An application of Napier's rule relating the three sides of a spherical right triangle gives

$$\cos(90° - L) = \cos C \cos(90° - E)$$

or

$$\sin L = \cos C \sin E \tag{6}$$

A corollary is also provided. The hour angle ZPM is the mean between the hour angles ZPX and ZPY. If H represents the hour angle ZPM, and we consider triangle ZPR (right angle at R), then through an application of another of Napier's rules,

$$\tan H = \tan C \sec E \tag{7}$$

In summary, to find the latitude, L, measure the solar altitudes, h_1 and h_2, noting the elapsed time, t between them. The Nautical Almanac provides the declination of the sun, D, on the date in question.

- A = length of XM = length of MY.
- B = complement of the declination of point M
- C = length of arc ZnR
- E = declination of R
- ζ = length of arc RM = declination of R - declination of M = $E - B$.
- L = the desired latitude
- H the hour angle ZnPM

Determine the latitude, L, and hour angle, H, by using the following equations in the order given, solving for A, B, C, ζ, E, and L in turn. If desired, H may also be determined.

- $\csc A = \sec D \csc \frac{t}{2}$.
- $\csc B = \csc D \cos A$
- $\sin C = \cos \frac{h_1+h_2}{2} \sin \frac{h_1-h_2}{2} \csc A$
- $\sec \zeta = \csc \frac{h_1+h_2}{2} \sec \frac{h_1-h_2}{2} \cos A \cos C$
- $E = B + \zeta$
- $\sin L = \cos C \sin E$
- $\tan H = \tan C \sec E$

These equations, derived in equations (1) through (7) above, are precisely the calculations implied by Bowditch's Rules. As Peirce points out, the elaborate sign rules in Bowditch can be replaced by the simpler approach of letting D and B be positive when latitude and declination are of the same name, but negative if contrary. Similarly, let ζ be positive if the zenith is nearer the elevated pole than it is to the point M, else let ζ be negative.