

An interesting feature resulting from the Astronomical/Geographic coordinates system on Ellipsoids

A recent thread started by M. Frank E. Reed (<https://navlist.net/triangle-equator-FrankReed-oct-2025-g57872>) keeps reminding us that Latitudes on Earth are defined by their Astronomical meaning and not by their Geocentric meaning.

- For reasons of best achievable depiction of the existing world all Maritime Charts referenced coordinates have long been based on various Ellipsoids (e.g. WGS84 from now on) vs. the formerly used Spherical reference[s].
- Nonetheless and for understandable reasons due to their [relative] simplicity Navigators have kept performing all applicable "Standard Navigation" computations on a Spherical Earth vs. an Ellipsoid shaped Earth.

This widely accepted *inconsistency* between such distinct references used by Navigators - i.e. Ellipsoids referenced Nautical Charts vs. Spherical computations - has curious and interesting implications albeit none of them being dramatic ones.

As an example among others, and as regards the *horizontal angular separation between two given points seen from a third one*, there are sizeable differences between their *accurately observed/recorded* values (e.g. through a *Cercle de Borda*) and their *computed* values assuming the Charts coordinates were referenced to a Sphere vs. an Ellipsoid.

In the following practical example, a Navigator in position A is assumed to closely measure the horizontal angular separation between 2 distant poles B and C. For the sake of simplicity, we assume that pole B is exactly North of the Navigator. The situation then all boils down to the Navigator actually measuring the Azimuth of the third pole C.

We then need to use some reliable and accurate computing mathematical tool adapted to Ellipsoids.

- One idea is to use Vincenty's Formulae. But they are not the exact mathematical tools required here since they compute departure azimuths of the [least distance] "Geodesics" towards targets. A quite different mathematical problem indeed, although these Formulae yield outstanding results for short distances (e.g. under 200 NM). Why not then invent some new and specific tool dedicated to directly computing Azimuths on Ellipsoids?
- Let us then rig the Navigator's local vertical axis with a rotating plan and move it until it contains the "target", at such time we can record its exact local Azimuth. This 3D computation mathematical tool exactly fits our specific purpose. It also offers the unique advantage of very easily accommodating all and any Targets/Observers altitudes changes, whereas Vincenty's Formulae are only applicable to the surface of their own referenced Ellipsoids.

Let us then work the [following example](#) : *Let's build three simple towers with flagpoles at the top that we can observe with sextants from a distance. The flagpoles will be positioned by GPS at the following locations:*

- A: 0°00' N, 50°00' W (0 m)
- B: 0°06' N, 50°00' W (0 m)
- C: 0°06' N, 49°52' W (0 m)

Computed results from A to C:

- (1) - Mid-Latitude straight line plan computation (1 NM = 1,852 meters): $D = 18,520.000\ m$ $Az = 053^{\circ}07'48.369''$
- (2) - "Full" Rhumb line computation on a sphere (1 NM = 1,852 meters): $D = 18,519.994\ m$ $Az = 053^{\circ}07'48.322''$
- (3) - Great Circle computation on a sphere (1 NM = 1,852 meters): $D = 18,519.994\ m$ $Az = 053^{\circ}07'48.177''$
- (4) - WGS84 Vincenty's : $D = 18,508.625\ m$ $Az\ 053^{\circ}18'52.566''$
- (5) - WGS84 3D Vertical Axis Method : $D = 18,508.625\ m$ $Az\ 053^{\circ}18'52.566''$

As expected we observe here that for such short distances involved and so near from the Equator:

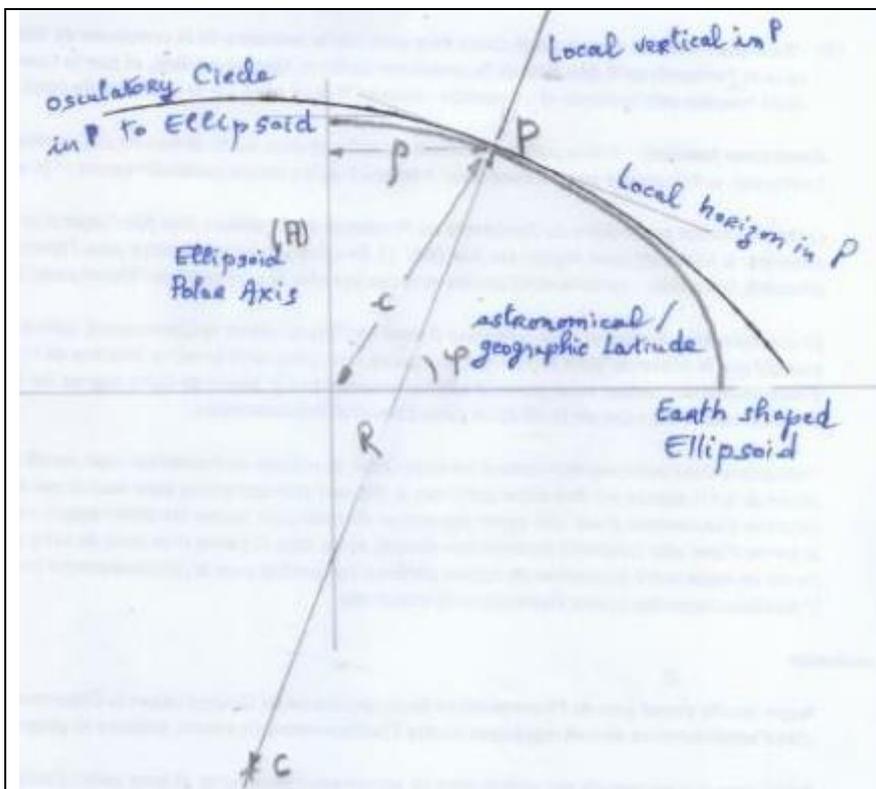
(1) , (2) and (3) yield extremely close results. And:

To the precision of the digits published (4) and (5) yield identical results.

An interesting challenge from M. Frank E. Reed here: "<https://navlist.net/triangle-equator-FrankReed-oct-2025-g57899>"

"Find yourself some targets for observation at ranges similar to the scenario in my original post at any latitude, calculate the sextant angles between targets using the plain latitudes and longitudes from GPS devices (or mapping intended to be used with GPS devices --meaning anything in the modern world), and you will discover comparably large errors in the angles [except at one latitude... anyone?]."

If any existing solution, we are to find Latitudes on the Ellipsoid were the length of a Degree of Longitude divided by the length of a Degree of Latitude exactly matches the same ratio for a Sphere. Let us then take a look at the following sketch:



From a point P on the surface of an Ellipsoid:

- The length of a degree at the astronomical/geographic Latitude ϕ is proportional to the length of the radius "R" of the Osculatory Circle to the Ellipsoid.
- The Length of degree of Longitude is similarly proportional to the length of its distance ρ to the North-South axis (P)

On a point of a Sphere, we know that this same ratio ρ/R is exactly equal to $\cos \phi$

Let us then compute these ρ/R ratios obtained on a Sphere and on WGS84 to see whether they are some existing Latitudes where they are equal.

Let's call $S = \rho/R \text{ (sph.)} / \rho/R \text{ (ell.)}$
 which simplifies into:
 $S = \cos \phi / \rho/R \text{ (ell.)}$

We obtain the following results for a few specific values of the Latitudes:

| Latitudes | Sphere $\rho/R \text{ (Sph.)} = \cos \phi$ | ρ km | WGS 84 R km | $\rho/R \text{ (ell.)}$ | S $S = \cos \phi / \rho/R \text{ (ell.)}$ |
|-----------|-----------------------------------------------|-----------|------------------|-------------------------|----------------------------------------------|
| 0° | 1.000 000 | 6,378.137 | 6,335.439 | 1.006 739 | 0.993 306 $= \underline{1 - 1/149.4}$ |
| 45° | 0.707 107 | 4,517.591 | 6,367.382 | 0.709 490 | 0.996642 |
| 54°46'50" | 0.576 710 | 3,686.578 | <u>6,378.137</u> | 0.578 002 | 0.997 763 |
| 70° | 0.342 020 | 2,187.928 | 6,392.033 | 0.342 290 | 0.999 212 |
| 88° | 0.034 899 | 223.342 | 6,399.515 | 0.034 900 | 0.999 992 |
| 89°59' | 0.000 291 | 1.862 | 6,399.594 | 0.000 291 | 1.000 000 |
| 90° | 0° | 0 | 6,399.584 | 0 | ----- |

We can therefore conclude that only very close to either the North or the South Pole can we expect for the difference between Azimuths of pole C seen from pole A calculated by either the 3D method or the Great Circle Method to stay minimal. Nonetheless the Rhumb Line results will differ significantly from them both.

As an example : from pole A at N89°54'00"/W090°00'00"/0 m to pole C at N89°52'00"/W000°00'00"/0 m

- (1) - "Full" Rhumb line computation on a sphere (1 NM = 1,852 meters): $D = 20,560.880 \text{ m}$ $Az = 100^{\circ}22'42.135''$
- (2) - Great Circle computation on a sphere (1 NM = 1,852 meters): $D = 18,519.994 \text{ m}$ $Az = 053^{\circ}07'48.597''$
- (3) - WGS84 Vincenty's : $D = 18,615.656 \text{ m}$ $Az = 053^{\circ}07'48.590''$
- (4) - WGS84 3D Vertical Axis Method : $D = 18,615.656 \text{ m}$ $Az = 053^{\circ}07'48.595''$

Only some quite [very] limited academic interest here.

Au Roc Saint Luc, le 20 Octobre 2025

Antoine M. Kermit Couëtte antoine.m.couette@club-internet.fr