An interesting feature resulting from the Astronomical/Geographic coordinates system used on Ellipsoids

A current thread started by M. Frank E. Reed (*https://navlist.net/triangle-equator-FrankReed-oct-2025-g57872*) keeps reminding us that Latitudes for Navigation are defined according to their Astronomical (*not* their Geocentric) meaning.

- To best describe and represent the existing world all Maritime Charts have long been using various ellipsoids (e.g. **WGS 84** from now on) instead of the spherical canvases formerly used to reference coordinates in the remote past.

- Nonetheless Navigators have stuck to their traditional long range Navigation computations tools using relatively simple spherical trigonometry formulae instead of their much more complex ellipsoidal trigonometry counterparts.

- This widely accepted *inconsistency* between 2 distinct references used by Navigators - Ellipsoids for Nautical Charts vs. Spheres for Navigation computations - shows curious results albeit none of them being a dramatic one.

An example among others: *there are sizeable differences between accurately observed and recorded values of the horizontal separation between two given points seen from a third one* (e.g. through *a Sextant or* a *Cercle de Borda*) *and the results derived from processing their Ellipsoid referenced coordinates through Spherical trigonometry.*

In the following practical example, a Navigator in position A is assumed to closely measure the horizontal angular separation between 2 distant flagpoles B and C. For the sake of simplicity, we assume that flagpole B is exactly North of the Navigator. Such situation then boils down to the Navigator actually measuring the Azimuth of flagpole C.

We then need some reliable and accurate mathematical tool adapted to Ellipsoids to compute real world observations.

- One first idea is to use the ***Vincenty's Formulae***. But these are not the exact mathematical tools required here since they compute departure azimuths of the [least distance] "Geodesics" towards targets. A quite different mathematical problem indeed, although these Formulae should yield outstanding results for short distances (e.g. < 200 NM). *Why not inventing some new and specific tool dedicated to directly computing Azimuths on Ellipsoids?*

- *Let us then rig the Navigator's local vertical axis with a rotating plan and move it until it contains the "target", at which time we can record its exact local Azimuth.* This ***3D computation mathematical tool*** exactly suits our specific needs. It is totally independent from any geodesic considerations whatsoever. It also has the unique advantage - not shared by the ***Vincenty's formulae*** - of very easily accommodating all and any Targets/Observers altitudes changes.

Let us then work the *following example* : *Let's build three simple towers with flagpoles at the top that we can observe with sextants from a distance. The flagpoles will be positioned by GPS at the following locations:*

A: 0°00' N, 50°00' W (0 m)   B: 0°06'N, 50°00' W (0 m)   C: 0°06' N, 49°52' W (0m)

Computed results from flagpole A to flagpole C:

(1) - Mid-Latitude straight line plan computation (1 NM = 1,852 meters): *D = 18,520.000 m Az = 053°07'48.369"*

(2) - "Full" Rhumb line computation on a sphere (1 NM = 1,852 meters): *D = 18,519.994 m Az = 053°07'48.322"*

(3) - Great Circle computation on a sphere (1 NM = 1,852 meters): *D = 18,519.994 m Az = 053°07'48.177"*

(4) - WGS84 ***Vincenty's Formulae*** : *D = 18,508.625 m Az = 053°18'52.566"*

(5) - WGS84 ***Vertical Axis 3D Method*** : *D = 18,508.625 m Az = 053°18'52.566"*

As expected we observe here that for such short distances involved and so near from the Equator:

- (1) , (2) and (3) yield extremely close results. And, interestingly enough:

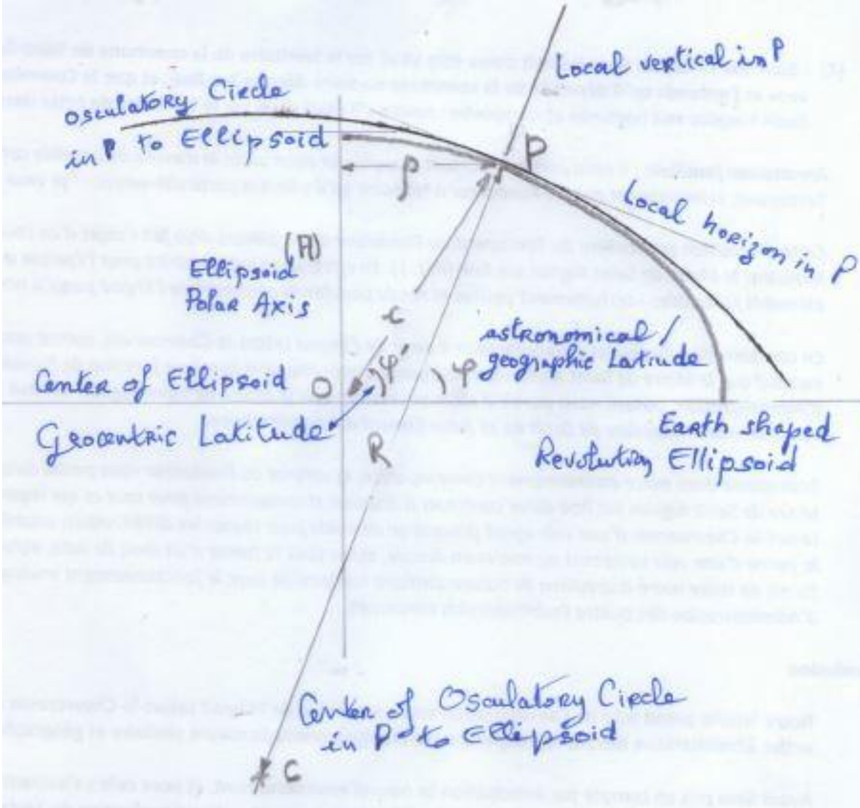
- To the precision of the digits published *(4) and (5) yield identical Azimuths,* ***in excess of 11' from (1), (2) and (3) Azimuths*****.**

****

An interesting challenge from M. Frank E. Reed here: "https://navlist.net/triangle-equator-FrankReed-oct-2025-g57899"

"Find yourself some targets for observation at ranges similar to the scenario in my original post at any latitude, calculate the sextant angles between targets using the plain latitudes and longitudes from GPS devices (or mapping intended to be used with GPS devices --meaning anything in the modern world), and you will discover comparably large errors in the angles [except at one latitude... anyone?]."

If any existing solution, we are to find Latitudes on the Ellipsoid where the length of a Degree of Longitude divided by the length of a Degree of Latitude exactly matches the same ratio for a Sphere. Let us then take a look at the following sketch:



From a point P on the surface of an Ellipsoid:

- The length of a degree at the astronomical/ geographic Latitude φ is proportional to the length of the radius "R" of the Osculatory Circle to the Ellipsoid.
- The Length of degree of Longitude is similarly proportional to the length of its distance ρ to the North-South axis (P)

On a point of a Sphere, we know that this same ratio ρ/R is exactly equal to cos φ

Let us then compare these ρ/R ratios obtained on a Sphere and on WGS84 to see whether they are some existing Latitudes where they are equal.

Let us define S = ρ/R (sph.) / ρ/R (ell.)

which simplifies into:

S = cos φ / ρ/R (ell.)

We then obtain the following results for a few specific values of the Latitudes: <u>Note</u>: S(0°) = 0.993 306 = 1 - 1/149.4

Latitudes	Sphere	*WGS 84*	*(♦LBL "EQ1")*		
	ρ/R (Sph.) = cos φ	ρ km	R km	ρ/R (ell.)	S = cos φ / ρ/R (ell.)
0°	1.000 000	*6,378.137*	6,335.439	1.006 739	0.993 306
45°	0.707 107	4,517.591	6,367.382	0.709 490	0.996642
54°46'50"	0.576 710	3,686.578	<u>6,378.137</u>	0.578 002	0.997 763
70°	0.342 020	2,187.928	6,392.033	0.342 290	0.999 212
88°	0.034 899	223.342	6,399.515	0.034 900	0.999 992
89°59'	0.000 291	1.862	6,399.594	0.000 291	1.000 000
90°	0°	0	6,399.584	0	*(1.000 000)*

We can therefore conclude that *only very close to either the North or the South Pole* can we expect that from flagpole A both flagpole C Azimuths calculated by the Great Circle and the 3D Methods should show some difference much smaller than the 11' observed at the Equator. Nonetheless the Rhumb Line results will likely differ very significantly from them both.

<u>As an example</u> : from *flagpole A at N89°54'00/W090°00'00/0 m* to *flagpole C at N89°52'00/W000°00'00/0 m*

(1) - "Full" Rhumb line computation on a sphere (1 NM = 1,852 meters): *D = 20,560.880 m Az = 100°22'42.135"*

(2) - Great Circle computation on a sphere (1 NM = 1,852 meters): *D = 18,519.994 m Az = 053°07'48.597"*

(3) - WGS84 ***Vincenty's Formulae*** : *D = 18,615.656 m Az = 053°07'48.590"* ***<u>almost identical to Az(2) this time</u>*** **.**

(4) - WGS84 ***Vertical Axis 3D Method*** : *D = 18,615.656 m Az = 053°07'48.595"* ***<u>almost identical to Az(2) this time</u>*** .

<u>Note</u>: In a different post (<u>https://navlist.net/triangle-equator-FrankReed-oct-2025-g57894</u>) the following formulae are given:

dy = 60 . dLat.[1 - (3/2)ε.cos(2L)] and dx = 60 . dLon.[1 + ε - (1/2) ε.cos(2L)].cos (L)

*<u>Solving for S = 1 from dx and dy</u> then boils down to solving for: ε [1 + cos(2L)] = 0. This **demonstrates** the above observed results since, except for Spheres as trivial solutions(ε = 0), only "cos(2L) = -1" is a solution, again implying L = +/- 90°.*

****

Ellipsoid paired "Twin" Spheres

<u>As earlier explained</u> the previous formulae *dy = 60 . dLat.[1 - (3/2)ε.cos(2L)]* and *dx = 60 . dLon.[1 + ε - (1/2) ε.cos(2L)].cos (L)* enable to compute to *<u>a good approximation</u>* distances in NM on an Ellipsoid between 2 points of either same Latitude (*dx*) or same Longitude (*dy*). *What is the scaling factor best matching **WGS 84** to compute such distances in kilometers then?*

We first need to have recourse to the following definition:

For any given Ellipsoid, its paired "*Twin Sphere*" has the same Equator to Pole meridian distance than this Ellipsoid.

e.g. with **WGS 84** having an Equator to Pole Meridian Distance of 10,001.965 728 km, *its **"Twin Sphere" SPH 84** has the following characteristics:*

Diameter = 40,007.862 912 km , R = 6,367.449 144 99 km , 1° = 111.132 952 533 km, altogether with:

1' = 1,852.215 875 55 km and 1' = 1.000 116 563 47 NM

Then, for ***SPH 84*** the correct scaling factor is *1' = 1,852.215 875 km*, so that the previous formulae can be rewritten in km as:
dy = 111.132 952 km . dLat.[1 - (3/2)ε.cos(2L)] and *dx = 111.132 952 km . dLon.[1 + ε - (1/2) ε.cos(2L)].cos (L).* And:
if dLat. and dLon. are given in arc minutes, then:
dy = 1,852.215 876 km . dLat.[1 - (3/2)ε.cos(2L)] and *dx = 1,852.215 876 km . dLon.[1 + ε - (1/2) ε.cos(2L)].cos (L).*

For various Latitudes the following updated spreadsheet compares the length of 1' of Longitude and 1' of Latitude obtained as follows: all data in blue printing are derived directly from **WGS 84**, while *all data in red printing are derived from **SPH 84**.*

Latitudes	***SPH 84***		**WGS 84** (♦LBL **"EQ1"**)							
	ρ/R=cos φ	*ρ km*	ρ km	1' (km)	*1' from dx*	R km	1' (km)	*1' from dy*	A=ρ/R(ell.)	S=cosφ/ A
0°	1.000 000	*6,367.449*	6,378.137	1.855 325	*1.855 321*	6,335.439	1.842 905	*1.842 901*	1.006 739	0.993 306
15°	0.965 926	*6,151.863*	6,162.189	1.792 508	*1.792504*	6,339.703	1.844 145	*1.844 149*	0.972 000	0.993 751
30°	0.866 025	*5,518.993*	5,528.257	1.608 105	*1.608 100*	6,351.177	1.847 151	*1.847 558*	0.870 403	0.994 971
45°	0.707 107	*4,510.021*	4,517.591	1.314 114	*1.314 106*	6,367.382	1.852 196	*1.852 216*	0.709 490	0.996 642
54°46'50"	0.576 710	*3,680.400*	3,686.578	1.072 382	*1.072 372*	6,378.137	1.855 325	*1.855 335*	0.578 002	0.997 763
70°	0.342 020	*2,184.261*	2,187.928	0.636 442	*0.636 433*	6,392.033	1.859 367	*1.859 352*	0.342 290	0.999 212
88°	0.034 899	*222.967*	223.342	0.064 967	*0.064 966*	6,399.515	1.861 543	*1.861 508*	0.034 900	0.999 992
89°	0.017 452	*111.501*	111.688	0.032 489	*0.032 488*	6,399.574	1.861 561	*1.861 525*	0.017 452	0.999 998
89°59'	0.000 291	*1.858*	1.862	0.000 542	*0.000 541*	6,399.594	1.861 566	*1.861 531*	0.000 291	1.000 000
90°	0.000 000	*0.000*	0.000	0.000 000	*0.000 000*	6,399.594	1.861 566	*1.861 531*	0.000 000	(1.000 000)

As a conclusion, when using the *dx* and *dy* formulae, ***SPH 84*** remains the best scaling factor matching **WGS 84**.

When measuring the lengths of 1' in Longitude and 1' in Latitude, the resulting *dx errors* range from 4 mm at the Equator to 0 mm at the Poles (of course …) while the *dy errors* range from 4 mm at the Equator to 35 mm at the Poles.

*Better than 5 * 10**-5 in relative values at all Latitudes.*

Let us work again our initial example with the use of formulae *dx* and *dy* hereabove, *which is deemed quite similar in its principle to using a **Meridional Parts Table**.*

Flagpole A at N89°54'00/W090°00'00/0 m and flagpole C at N89°52'00/W000°00'00/0 m

Computed results from flagpole A to flagpole C

Difference in Latitude : 6' , hence *dy* = 6 * 1.842 905 km = 11.057 430 km

Difference in Longitude : 8' , hence *dx* = 8 * 1.855 325 km = 14.842 600 km

*Plane triangle solution through the **R→P** function: D = 18.508 634 km departure Azimuth 053°18'52.745"*

vs. **Vertical Axis 3D Method / Vincenty's Method :** D = 18.508 625 km Az : 053°18'52.566".

Conclusion: Such "*[Refined] Meridional Parts*" computed through the quite simple *dx* and *dy* formulae *fitted to the **SPH 84 Sphere*** enable to compute almost perfect results on the **WGS 84 Ellipsoid** for such small distances close from the Equator.

Au Roc Saint Luc en Pissotte (France), le 31 Octobre 2025 Antoine M. Kermit Couëtte antoine.m.couette@club-internet.fr