An all-haversine method for manual calculation of zenith distance, without multiplication

Lars Bergman, 18 November 2015

As shown before (by Hanno and Greg)

$\text{hav}\, z = N + (1 - Q)\text{hav}(lha)$

where $N = \text{hav}(lat - dec)$, $P = \text{hav}(lat + dec)$ and $Q = N + P$

The multiplication of two 4-figure numbers in the second term is the error-prone part of the solution. Is it possible to get rid of this multiplication? By expanding above expression we get

$\text{hav}\, z = \text{hav}(lat - dec) + \text{hav}(lha) - \text{hav}(lat - dec)\text{hav}(lha) - \text{hav}(lat + dec)\text{hav}(lha)$

After some manipulation we'll find that the product of two haversines is

$$\text{hav}(x)\text{hav}(y) = \frac{1}{2}\left(\text{hav}(x) + \text{hav}(y) - \frac{\text{hav}(x+y) + \text{hav}(x-y)}{2}\right)$$

Thus we get, after some further manipulation

$$\text{hav}\, z = \frac{1}{2}\left[\text{hav}(lat - dec) - \text{hav}(lat + dec) + \frac{\text{hav}(lat - dec + lha) + \text{hav}(lat - dec - lha) + \text{hav}(lat + dec + lha) + \text{hav}(lat + dec - lha)}{2}\right]$$

By defining an additional four parameters

$Np = \text{hav}(lat - dec + lha)$
$Nn = \text{hav}(lat - dec - lha)$
$Pp = \text{hav}(lat + dec + lha)$
$Pn = \text{hav}(lat + dec - lha)$

we get the simple expression $\text{hav}(z) = \frac{1}{2}\left(N - P + \frac{Np + Nn + Pp + Pn}{2}\right)$

Now the difficult multiplication is replaced by two simple "divide by 2", but the price we pay is the calculation of a few more angles and looking up their haversines; nothing is for free!

An example using Hanno's table of $10^4\text{hav}(x)$

lat	59° 18'		
dec	-19 13		
	78 31	*N*=4004.5	
lha	27 45		
	106 16		*Np*=6401
	50 46		*Nn*=1838
	40 5	*P*=1174.5	
	67 50		*Pp*=3113
	12 20		*Pn*= 115
		2830 (diff)	11467 (sum)
			5734 (div by 2)
			2830 (*N-P*)
			8564 (add)
			4282 (div by 2) giving z=81° 45' and thus altitude = 8° 15'