COMPACT DATA FOR NAVIGATION AND ASTRONOMY

for the years
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3. The true azimuth (Z) is calculated from

$$X = \cos\phi \sin DEC - \sin\phi \cos DEC \cos LHA$$
$$Y = -\cos DEC \sin LHA$$
$$A = \tan^{-1}(Y/X)$$

If $X < 0$ then $Z = A + 180°$
If $X > 0$ and $Y < 0$ then $Z = A + 360°$
Else $Z = A$.

To avoid division by zero at the E and W cardinal points, which may produce an error in the calculator, add a very small number to the denominator X before division (eg $1{\cdot}0 \times 10^{-10} \equiv 1{\cdot}0\text{E}{-10}$).

Many calculators have the facility to convert directly from rectangular coordinates (x, y) to polar coordinates (r, θ). Set $x = X$, $y = Y$ and convert to polar coordinates. Then $Z = \theta$.

7.3 *Position from intercept and azimuth using position lines* The position line for an observation is determined from the intercept $p = H_o - H_c$ and the azimuth Z as follows:

1. Make an initial estimate of the position (λ_f, ϕ_f) at the time of fix (t_f).

2. Determine the position (λ, ϕ) at the time of observation (t) by dead reckoning (DR) from the time of fix. For example, if the bearing (or course) T and the speed V (in knots) remain constant

then $\lambda = \lambda_f + (t - t_f)(V/60) \sin T/\cos\phi_f$
$\phi = \phi_f + (t - t_f)(V/60) \cos T$

where time is measured in hours.

3. Using the position at the time of observation (λ, ϕ) and the formulae in section 7.2 calculate the altitude H_c and azimuth Z. Calculate the intercept

$$p = H_o - H_c$$

4. The equation of a position line with origin at the estimated position at the time of fix (λ_f, ϕ_f) is given by

$$x \sin Z + y \cos Z = p$$

where the x-axis is parallel to lines of constant latitude, positive to the east, and the y-axis is along the meridian, positive to the north. If p is expressed in nautical miles (nm) then x and y will be in nautical miles also. Then the position line may be drawn on a chart in the standard way using the intercept and azimuth p, Z.

This procedure is repeated for each observation. If the errors are small the position will lie near the points of intersection of the position lines so that at least two position lines are required to obtain a fix.

7.4 *Position from intercept and azimuth by calculation* Alternatively the position of the fix may be calculated directly from two or more sextant observations, by using the method of least squares as follows:

If p_1, Z_1 are the intercept and azimuth of the first observation, p_2, Z_2 of the second observation and so on, form the quantities

$$
\begin{aligned}
A &= \cos^2 Z_1 + \cos^2 Z_2 + \cdots \\
B &= \cos Z_1 \sin Z_1 + \cos Z_2 \sin Z_2 + \cdots \\
C &= \sin^2 Z_1 + \sin^2 Z_2 + \cdots \\
D &= p_1 \cos Z_1 + p_2 \cos Z_2 + \cdots \\
E &= p_1 \sin Z_1 + p_2 \sin Z_2 + \cdots \\
F &= p_1^2 + p_2^2 + \cdots
\end{aligned}
$$

As a check verify that $A + C = n$ where n is the total number of observations included in the solution.

Calculate $G = AC - B^2$

Then an improved estimate of the position at the time of fix is

$$\lambda_f + d\lambda, \quad \phi_f + d\phi$$

where $d\lambda = (AE - BD)/(G \cos \phi_f)$ and $d\phi = (DC - EB)/G$

Additional observations may be included in the solution by simply adding the extra terms to A, B, C, D and E and calculating $d\lambda$ and $d\phi$ again. Similarly, observations may be rejected by subtracting the terms that were originally added to A, B, C, D and E and calculating $d\lambda$ and $d\phi$ again.

If the initial position λ_f, ϕ_f is not well known it will be necessary to iterate until a consistent solution is found. At each iteration λ_f and ϕ_f are updated to the latest calculated position. Indeed it is possible to start from any position on the Earth, and, provided that λ_f is kept in the range $-180°$ to $+180°$ and ϕ_f in the range $-90°$ to $+90°$, the solution in most cases will begin to converge after a few iterations.

7.5 *Estimated position error* If three or more position lines are obtained an estimate of the error in position may be calculated. The standard deviation of the estimated position σ in nautical miles is given by

$$\sigma = 60(S/(n-2))^{\frac{1}{2}}$$

where $S = F - D\,d\phi - E \cos \phi_f\, d\lambda$

The standard deviations σ_λ, σ_ϕ in longitude and latitude are given by

$$\sigma_\lambda = \sigma(A/G)^{\frac{1}{2}} \quad \sigma_\phi = \sigma(C/G)^{\frac{1}{2}}$$

In general as the number of observations increases the error in the estimated position decreases. Statistical theory shows that the estimated position has a probability P of lying within a confidence ellipse which is specified by the lengths of its axes a and b and the azimuth θ of the a-axis, where

$$
\begin{aligned}
\tan 2\theta &= 2B/(A - C) \\
a &= \sigma k/(n/2 + B/\sin 2\theta)^{\frac{1}{2}} \\
b &= \sigma k/(n/2 - B/\sin 2\theta)^{\frac{1}{2}}
\end{aligned}
$$

and $k = (-2 \log_e(1 - P))^{\frac{1}{2}}$ is a scale factor.

Values of the scale factor k for selected values of P are given in the table below.

Probability, P	0·39	0·50	0·75	0·90	0·95
Scale factor, k	1·0	1·2	1·7	2·1	2·4

Normally a confidence level of 95% is chosen, that is $P = 0{\cdot}95$.

The shape of the confidence ellipse depends only upon n and the distribution of the observations in azimuth; whilst the size of the ellipse, apart from the scale factor, depends upon the errors of observation. The method assumes that the observations have equal weight. The best results will be obtained when the observations are equally spaced in azimuth. In such cases the effect of systematic errors on the final calculated position will be minimised.

7.6 *Plotting position lines and the confidence ellipse with a personal computer*
Personal computers usually have facilities for plotting either on a TV monitor screen or on some other plotting device. They normally use the principle that a cursor or pen is moved from a point A with rectangular coordinates x_A, y_A to a point B with rectangular coordinates x_B, y_B and a straight line is drawn joining A and B in the process. A curve, such as an ellipse, is drawn by approximating it to a succession of short straight lines; the curve looks smooth to the eye provided that the lines are short enough.

The origin for plotting is chosen initially to be the estimated position of the fix with the x-axis along the line of latitude, positive to the east, and the y-axis along the meridian, positive to the north.

The position lines and confidence ellipse are drawn inside a square of length 20 nautical miles centred on the estimated position at the time of fix with sides parallel to the x and y axes.

To draw the position line for intercept p (in nautical miles) and azimuth Z first find the two points where this line crosses the sides of the square. Set $x = \pm 10$. Then $y = (p \mp 10 \sin Z)/\cos Z$. The line crosses the side of the square if $-10 \leqslant y \leqslant 10$. Similarly set $y = \pm 10$. Then $x = (p \mp 10 \cos Z)/\sin Z$. The line crosses the sides of the square if $-10 \leqslant x \leqslant 10$.

The confidence ellipse is centred at the calculated position of the fix and is drawn by connecting up the rectangular coordinates (x, y) of a series of points that lie on the ellipse, which is specified by the previous calculated values of a, b and θ. The values of x and y are calculated from

$$x = a \cos \alpha \sin \theta - b \sin \alpha \cos \theta + 60\, d\lambda \cos \phi_f$$
$$y = a \cos \alpha \cos \theta + b \sin \alpha \sin \theta + 60\, d\phi$$

where α takes the value from 0° to 360° in steps of, say, 15°. By trial and error find the optimum step length for α below which there is no noticeable improvement in the smoothness of the ellipse which the computer draws.

At the next iteration move the origin to the latest calculated position of the fix.

Example On 1986 June 15 at $21^h\,00^m\,00^s$ GMT the dead reckoning (DR) position of a ship was W 15° 12′, N 32° 30′. At $17^h\,30^m\,45^s$ the sextant altitude of the lower limb of the Sun was 29°·9857. At $18^h\,15^m\,24^s$ the sextant altitude of the lower limb of the

Moon was 56°·9752. After sunset the sextant altitude of *Vega* (No 49) at $20^h\ 12^m\ 20^s$ was 21°·4853, and of *Dubhe* (No 27) at $20^h\ 23^m\ 15^s$ was 55°·2767. The ship maintained a constant track and speed of 315° and 12 knots, respectively. Calculate the position of the ship at $21^h\ 00^m\ 00^s$, assuming that height of eye above the horizon is 6 m and the sextant index error is zero, using the formulae and methods in section 7.

The time of fix $t_f = 21^h{\cdot}0000$. Adopting the DR position as the estimated position at the time of fix then $\lambda_f = -15°{\cdot}2000$, $\phi_f = +32°{\cdot}5000$. Intermediate values obtained in the calculation are shown in the table.

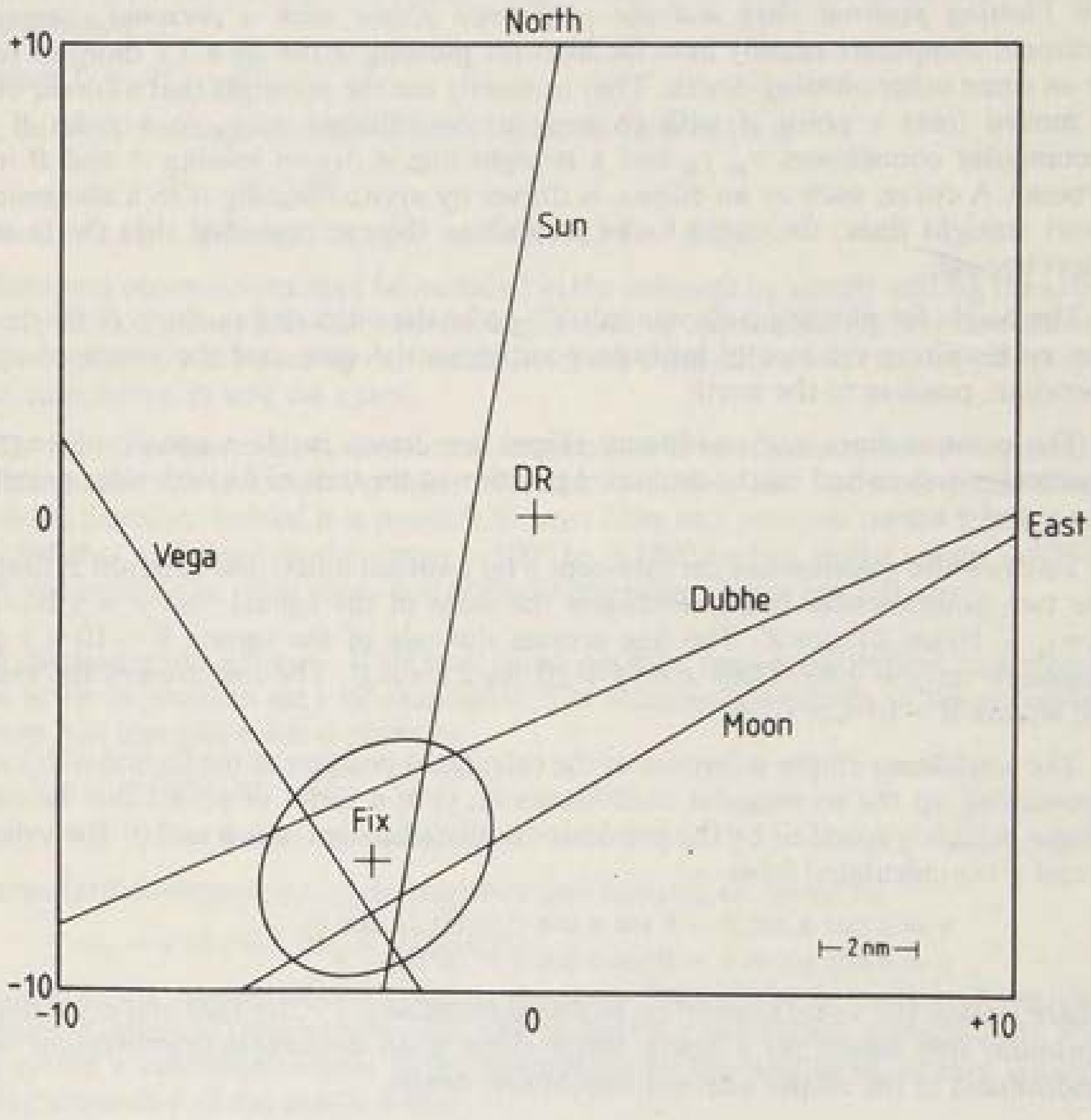

Figure: The four position lines shown correspond to the first iteration in the numerical example on page xxiii. The chart is centred at the DR position and the length of the side is 20 nautical miles (nm). The confidence ellipse is drawn for the 95% probability level.

Body	Sun	Moon	*Vega*	*Dubhe*
t	$17^h\ 30^m\ 45^s$	$18^h\ 15^m\ 24^s$	$20^h\ 12^m\ 20^s$	$20^h\ 23^m\ 15^s$
	17·5125	18·2567	20·2056	20·3875
λ (section 7.3)	−14·6152	−14·7400	−15·0668	−15·0973
ϕ	+32·0068	+32·1120	+32·3876	+32·4134
GHA (sections 2, 3, 5)	82·5829	358·7759	287·7705	43·9070
DEC	+23·3211	+3·3713	+38·7668	+61·8305
HP or *S*	0·2629	0·9605		
LHA (section 7.2)	67·9677	344·0359	272·7037	28·8097
Z	280·1973	149·1893	56·8311	336·4710
H_c	30·1285	57·5859	21·4970	55·2592
H_S (section 7.1)	29·9857	56·9752	21·4853	55·2767
D_h	0·0718	0·0718	0·0718	0·0718
I	0·0000	0·0000	0·0000	0·0000
H	29·9139	56·9034	21·4135	55·2049
R	0·0282	0·0106	0·0413	0·0113
PA	0·0021	0·5245		
Oblateness of Earth (3.3)		−0·0026		
S	0·2629	0·2617		
H_o	30·1507	57·6765	21·3722	55·1937
p (section 7.3)	+0·0222	+0·0906	−0·1248	−0·0656
	(+1·3 nm)	(+5·4 nm)	(−7·5 nm)	(−3·9 nm)

First iteration

$A = 1{\cdot}9089$	$B = -0{\cdot}5222$	$C = 2{\cdot}0911$
$D = -0{\cdot}2023$	$E = -0{\cdot}0537$	$F = 0{\cdot}0286$
$a = 2{\cdot}756$	$b = 2{\cdot}101$	$\theta = 40{\cdot}0547$
$P = 95\%$	$n = 4$	$\sigma = 1{\cdot}3651$
$\lambda_f = 15{\cdot}2664$		$\phi_f = 32{\cdot}3787$
= W 15° 16′·0		= N 32° 22′·7

The figure shows the four position lines and the confidence ellipse after the first iteration. The coordinates of the points where the position lines cross the sides of the square are

Sun	(−3·2, −10);	(0·4, 10)
Moon	(−6·2, −10);	(10, −0·4)
Vega	(−2·4, −10);	(−10, 1·6)
Dubhe	(−10, −8·6);	(10, 0·1)

Second iteration

$A = 1{\cdot}9083$	$B = -0{\cdot}5229$	$C = 2{\cdot}0917$
$D = -0{\cdot}0005$	$E = 0{\cdot}0016$	$F = 0{\cdot}0011$
$a = 2{\cdot}804$	$b = 2{\cdot}136$	$\theta = 40{\cdot}0284$
$P = 95\%$	$n = 4$	$\sigma = 1{\cdot}3883$
$\lambda_f = 15{\cdot}2655$		$\phi_f = 32{\cdot}3787$
= W 15° 15′·9		= N 32° 22′·7