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ON A NEW METHOD OF CORRECTING LUNAR DISTANCES FOR PARALLAX AND REFRACTION.

By WILLIAM CHAUVENET,

PROFESSOR OF MATHEMATICS IN THE U. S. NAVAL ACADEMY.

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(*Continued from page* 24.)

I.

The observation is supposed to give the apparent distance and apparent altitudes of the two objects; but if the latter cannot be observed, they must, in order to apply the present method, be previously computed by the known rules. Taking at once the most general case, namely, that in which the object observed with the moon also has parallax, let us call this object "the sun." Our formulas will require no change for a planet, and for a star no other change than making its parallax zero.

Let, then,

d = apparent distance of moon's and sun's centers. (See Section IV.)

h = moon's apparent altitude.

H = sun's " "

d_1, h_1, H_1, the distance and altitudes referred to that point of the earth's axis which lies in the vertical of the observer, which point (after Bessel) we shall distinguish as the point O.

We have the known fundamental relation,

$$\frac{\cos d_1 - \sin h_1 \sin H_1}{\cos h_1 \cos H_1} = \frac{\cos d - \sin h \sin H}{\cos h \cos H}$$

and if

$$m = \frac{\sin h_1 \sin H_1}{\sin h \sin H}, \qquad n = \frac{\cos h_1 \cos H_1}{\cos h \cos H},$$

then

$$\cos d - \cos d_1 = (1-n)\cos d - (m-n)\sin h \sin H. \qquad (1)$$

Let

$$\varDelta d = d_1 - d, \qquad \varDelta h = h_1 - h, \qquad \varDelta H = H - H_1,$$

then

$$\cos d - \cos d_1 = 2\sin\tfrac{1}{2}\varDelta d \sin(d + \tfrac{1}{2}\varDelta d)$$

$$n = \frac{\cos(h+\Delta h)\cos(H-\Delta H)}{\cos h \cos H}$$

$$= \left(1 - \frac{2\sin\frac{1}{2}\Delta h \sin(h+\frac{1}{2}\Delta h)}{\cos h}\right) \times \left(1 + \frac{2\sin\frac{1}{2}\Delta H \sin(H-\frac{1}{2}\Delta H)}{\cos H}\right),$$

$$1-n = \frac{2\sin\frac{1}{2}\Delta h \sin(h+\frac{1}{2}\Delta h)}{\cos h} - \frac{2\sin\frac{1}{2}\Delta H \sin(H-\frac{1}{2}\Delta H)}{\cos H_1}$$
$$+ \frac{4\sin\frac{1}{2}\Delta h \sin\frac{1}{2}\Delta H \sin(h+\frac{1}{2}\Delta h)\sin(H-\frac{1}{2}\Delta H)}{\cos h \cos H}.$$

Also, observing the relations

$$\sin h_1 \cos h = \tfrac{1}{2}[\sin(2h+\varDelta h) + \sin\varDelta h]$$
$$\cos h_1 \sin h = \tfrac{1}{2}[\sin(2h+\varDelta h) - \sin\varDelta h]$$
$$\sin H_1 \cos H = \tfrac{1}{2}[\sin(2H-\varDelta H) - \sin\varDelta H]$$
$$\cos H_1 \sin H = \tfrac{1}{2}[\sin(2H-\varDelta H) + \sin\varDelta H]$$

we find

$$m - n = \frac{\sin h_1 \cos h \sin H_1 \cos H - \cos h_1 \sin h \cos H_1 \sin H}{\sin h \cos h \sin H \cos H}$$

$$= \frac{\sin\Delta h \sin(2H-\Delta H) - \sin\Delta H \sin(2h+\Delta h)}{2\sin h \cos h \sin H \cos H}$$

If, then, we put

$$A_1 = \frac{2\sin\frac{1}{2}\Delta h \sin(h+\frac{1}{2}\Delta h)\cos d}{\cos h}$$

$$B_1 = -\frac{\sin\Delta h \sin(2H-\Delta H)}{2\cos h \cos H}$$

$$C_1 = -\frac{2\sin\frac{1}{2}\Delta H \sin(H-\frac{1}{2}\Delta H)\cos d}{2\cos H}$$

$$D_1 = \frac{\sin\Delta H \sin(2h+\Delta h)}{2\cos h \cos H}$$

the equation (1) becomes

$$2\sin\tfrac{1}{2}\Delta d \sin(d+\tfrac{1}{2}\Delta d) = A_1 + B_1 + C_1 + D_1 - A_1 C_1 \sec d. \qquad (2)$$

This *rigorous* formula may be adapted for practical use in several ways requiring auxiliary tables. I proceed to give the transformation which appears to require the fewest and simplest tables.

II.

If the terms of (2) are reduced to seconds, we shall have

$$\Delta d \sin(d+\tfrac{1}{2}\Delta d) = A_1 + B_1 + C_1 + D_1 - A_1 C_1 \sin 1'' \sec d \qquad (3)$$

in which

$$A_1 = \frac{\Delta h}{\cos h} \cdot \sin(h + \tfrac{1}{2}\Delta h) \cos d$$
$$B_1 = -\frac{\Delta h}{\cos h} \cdot \frac{\sin(2H - \Delta H)}{2\cos H}$$
$$C_1 = -\frac{\Delta H}{\cos H} \cdot \sin(H - \tfrac{1}{2}\Delta H) \cos d$$
$$D_1 = \frac{\Delta H}{\cos H} \cdot \frac{\sin(2h + \Delta h)}{2\cos h}.$$

Let

p = moon's horizontal parallax reduced to the point O. (See Section III.)

r = moon's refraction.

P, R, the same quantities for the sun; then

$$\Delta h = p\cos(h - r) - r$$
$$\Delta H = R - P\cos(H - R).$$

The neglect of R in the term $P\cos(H - R)$ produces an error altogether inappreciable in practice; but the error produced by omitting r in the term $p\cos(h - r)$ may amount to 1″, and we shall therefore take

$$\cos(h - r) = \cos h + \sin r \sin h$$
$$\Delta h = p\cos h - r + p\sin r\sin h$$
$$= (p\cos h - r)\left(1 + \frac{p\sin r\sin h}{p\cos h - r}\right).$$

If we develop the last term, and put

$$k = r\tan h$$

we shall have, designating the term by K,

$$K = \frac{p\sin r\sin h}{p\cos h - r} = k\sin 1''\left(1 + \frac{k}{p\sin h}\right)$$

in which p may be taken at its mean value; and since k and h decrease together, it will be found that K is nearly constant, its maximum being .000296, and its minimum .000285. A wider range will be admitted if we allow for the variations of the barometer and thermometer, and of p; but without here entering into more details, it will suffice to state that the error of the value

$$K = .00029$$

is always less than .00006 so long as $h > 5°$, and the formula

$$\Delta h = (p\cos h - r)(1 + K)$$

gives Δh within 0″.05 at a mean state of the air, and within 0″.2 in all cases.

Let now

$$r' = \frac{r}{\cos h}, \quad R' = \frac{R}{\cos H}.$$

The quantities r' and R' will be given by a "Refraction Table for Lunars," which with the argument *apparent altitude* will give the refraction divided by the cosine of the altitude, and will be arranged precisely like the ordinary tables of refractions. The corrections for the barometer and thermometer may be arranged as usual in nautical tables, with the arguments *height of barometer* (or *thermometer*) and *apparent altitude*; or, which is preferable, with the refraction itself instead of the altitude, for with the latter arrangement the same table will serve to give the correction either of r or of r'.

These quantities then being substituted, the corrections of the apparent altitudes become

$$\Delta h = (p - r')(1 + K)\cos h$$
$$\Delta H = (R' - P)\cos H$$

and the terms of (3) become

$$A_1 = (p - r')(1 + K)\sin(h + \tfrac{1}{2}\Delta h)\cos d$$
$$B_1 = -(p - r')(1 + K)\frac{\sin(2H - \Delta H)}{2\cos H}$$
$$C_1 = -(R' - P)\sin(H - \tfrac{1}{2}\Delta H)\cos d$$
$$D_1 = (R' - P)\frac{\sin(2h + \Delta h)}{2\cos h}.$$

The term $A_1 C_1 \sin 1'' \sec d$ is very small, its maximum being only about 1″. It is easy to obtain an approximate expression for it, and to combine it with the term A_1; for in so small a term we may take

$$C_1 = -R'\sin H\cos d = -k'\cos d$$

where $k' = R\tan H$; and without sensible error in most cases we may take $k'\sin 1'' = K$, so that

$$C_1 \sin 1'' \sec d = -K$$

and

$$A_1 - A_1 C_1 \sin 1'' \sec d = (p - r')(1 + K)^2 \sin(h + \tfrac{1}{2}\Delta h)\cos d.$$

The error of this evaluation of the term $A_1 C_1 \sin 1'' \sec d$ is produced chiefly by the neglect of P, and is therefore appreciable only in the case of the planet *Venus*. If we suppose the extreme case in which P, $p - r'$, and H are *all* at their maximum values, the error in this term is

$$0''.44 \cos d$$

and since the equation (3) is yet to be divided by $\sin d$, the final error in the distance is

$$0''.44 \cot d$$

and can amount to 1″ only when $d < 24°$. Moreover, the error is of the less importance in the case of *Venus*, because much less than the probable error of observation arising from an imperfect *bisection* of the planet's disc in the feeble telescope of the sextant.

Now let

$$\left.\begin{aligned} A &= (1 + K)^2 \cdot \frac{\sin(h + \frac{1}{2}\Delta h)}{\sin h} \\ B &= (1 + K) \cdot \frac{\sin(2H - \Delta H)}{\sin 2H} \\ C &= \frac{\sin(H - \frac{1}{2}\Delta H)}{\sin H} \\ D &= \frac{\sin(2h + \Delta h)}{\sin 2h} \\ &\text{and} \\ A' &= (p - r')\,A\sin h\cot d \\ B' &= -(p - r')\,B\sin H\,\mathrm{cosec}\,d \\ C' &= -(R' - P)\,C\sin H\cot d \\ D' &= (R' - P)\,D\sin h\,\mathrm{cosec}\,d \end{aligned}\right\} \quad \textbf{(A)}$$

then our formula (3) becomes

$$\varDelta d . \frac{\sin (d + \frac{1}{2} \Delta d)}{\sin d} = A' + B' + C' + D'.$$

Developing the first member, it becomes

$$\varDelta d \left(1 + \frac{2 \sin \frac{1}{2} \Delta d \cos (d + \frac{1}{2} \Delta d)}{\sin d}\right)$$

so that if we put

$$x = - \frac{\Delta d^2 \sin 1'' \cos (d + \frac{1}{2} \Delta d)}{\sin d}$$

or, with sufficient accuracy,

$$x = - \varDelta d^2 \sin 1'' \cot d \qquad \textbf{(B)}$$

we have finally

$$\varDelta d = A' + B' + C' + D' + x. \qquad \textbf{(C)}$$

The logarithms of A, B, C, and D can be given in extremely simple tables, requiring little or no interpolation, the arguments for log A and log D being $p - r'$ and h, and those for log B and log C being $R' - P$ and H. A', B', C', and D' may then be computed with the greatest ease. The value of x can be given in a small table with the arguments $\varDelta d$ and d, the table being first entered with the approximate value of $\varDelta d = A' + B' + C' + D'$.

The advantages of the preceding process are conceived to be, — 1st. The formula is almost rigorously exact, representing the correction of distance in all practical cases within 1″; 2d. The logarithmic computation is simple and brief; 3d. *The tabulated logarithms require no correction for the height of the barometer and thermometer.* In no one of the approximative methods in use are these features combined. Those which are based upon accurate formulas either require troublesome computations, or are shortened by the use of tables in which a mean refraction is used, and no ready method is given for correcting the logarithms in these tables for the actual state of the air. Such for the most part are Bowditch's methods. It would hardly be necessary to allude to those which are *not* based upon accurate formulas, were it not that one of this character has been adopted in a comparatively recent work of great merit in most respects, Raper's *Practice of Navigation.* The approximative method employed in that work is one received from Mendoza Rios, apparently without a very critical examination; in favorable circumstances, and particularly in low latitudes, it may be so applied as to be sufficiently accurate, but in high latitudes cases are common in which the error in the distance is 10″, and in the *extreme* case the error is 50″.*

If we compare our method with the shortest of the rigorous processes of spherical trigonometry, we find, — 1st. It is simpler in the logarithmic computation, requiring only four-decimal, or at most five-decimal logarithms. It is also an important simplification for the practical navigator, that the distance and altitudes are not required to be combined (to form, for example, their half sum, etc.), previously to referring to the tables, as in almost every other method, approximative or rigorous. 2d. It separates the principal corrections for the moon and sun, the principal correction for the moon being $A' + B'$, and that for the sun being $C' + D'$. The advantage of this separation appears in the method to be given for computing the correction for contraction of the moon's and sun's semidiameters by refraction. (Section IV.)

III.

*Correction for the Compression of the Earth.** — In the preceding investigation d_1, h_1, H_1, represent the distance and altitudes referred to the point O. (Section I.) This reference may be made in the case of the moon by employing a horizontal parallax equal to her equatorial horizontal parallax increased in the ratio $\frac{a_1}{a}$, a denoting the equatorial radius of the earth, and a_1 the distance of the observer from the point O; which distance is the normal of the spheroid, and is expressed by

$$a_1 = \frac{a}{\sqrt{(1 - \varepsilon^2 \sin^2 \varphi)}}$$

where ε = eccentricity of the meridian,
φ = geodetic latitude.

This process is subject to a slight theoretical error, the amount of which will presently be estimated.

If we denote by $-a i$ the distance from the center of the earth to the point O, and put

π = moon's equatorial horizontal parallax,
ϱ = distance of the moon from the center of the earth,
δ = moon's geocentric declination,
d' = angular distance of the moon and sun referred to the center of the earth,
π_1, ϱ_1, δ_1, d_1, = the same quantities referred to the point O,
$\varDelta$ = sun's declination,
α = difference of right ascensions of the moon and sun,

* The development of $\varDelta d$ in series as far as the terms of the second order is

$$\varDelta d = - \varDelta h \cos q + \varDelta H \cos Q + \frac{\Delta h^2}{2} \cdot \frac{\sin^2 q}{\tan d}$$
$$+ \varDelta h . \varDelta H . \frac{\sin q \sin Q}{\sin d} + \frac{\Delta H^2}{2} \cdot \frac{\sin^2 Q}{\tan d}$$

in which q and Q denote the angles included between d and the zenith-distances of the moon and sun respectively. In the method referred to in the text, the formula employed is equivalent to

$$\varDelta d = - \varDelta h \cos q + \varDelta H \cos Q + \frac{\Delta h^2}{2} \cdot \frac{\sin^2 q}{\tan d}$$

so that the error E in the distance is

$$E = - \left(\varDelta h . \varDelta H . \frac{\sin q \sin Q}{\sin d} + \frac{\Delta H^2}{2} \cdot \frac{\sin^2 Q}{\tan d} \right).$$

If $h = 10°$, $H = 10°$, $d = 40°$, $p = 60'$, (which is far from being an extreme case,) we find $E = -10''.4$. In the extreme case to which, according to the arrangement of Raper's tables and rules, the method may be extended, — namely, $h = 5°$, $H = 5°$, $d = 13°$, $p = 61' \ 30''$, — the refraction being at a mean, we find $E = -43''$; and since we may suppose the refraction to be increased by as much as one sixth of its amount, we may even have $E = -50''$.

Bowditch's "First Method" is more accurate as to the formula, including one term more of the above development, but, being adapted for use only upon the supposition of a mean state of the air, is subject to even greater errors than the method just examined.

* For the sake of completeness I have thought it proper to treat of this correction fully; but the method followed, it need hardly be observed, is not new in principle, although it is not at present in use among navigators.

then we have the known formulas

$$a i = \frac{a\,\epsilon^2 \sin\varphi}{\sqrt{(1-\epsilon^2 \sin^2\varphi)}}$$

$$\left.\begin{aligned} \varrho_1 \cos\delta_1 &= \varrho \cos\delta \\ \varrho_1 \sin\delta_1 &= \varrho \sin\delta + a i \end{aligned}\right\} \qquad (4)$$

whence, very nearly,

$$\varrho_1 = \varrho + a i \sin\delta \qquad (5)$$

$$\sin\pi_1 = \frac{a_1}{\varrho_1} = \frac{a_1}{\varrho}\left(1 + \frac{a i \sin\delta}{\varrho}\right)^{-1}$$

$$= \frac{a_1}{a} \sin\pi\,(1 - i \sin\pi \sin\delta + \&c.)$$

or, with extreme accuracy,

$$\pi_1 = \pi \,.\, \frac{a_1}{a} - \frac{\epsilon^2 \sin^2\pi \sin\varphi \sin\delta}{\sin 1''}.$$

The maximum value of the last term is only 0″.2, so that in the present application we may take

$$\pi_1 = \pi \,.\, \frac{a_1}{a}$$

and the correction of π,

$$\pi_1 - \pi = \pi \,.\, \frac{a_1 - a}{a}$$

may be given in a small table with the arguments φ and π. The similar correction of the sun's or a planet's parallax is insensible in practice.

If, then, in the computation of (**A**), (**B**), and (**C**), we employ for p the value $p = \pi_1$ we obtain d_1. To reduce finally to the center of the earth, we have

$$\left.\begin{aligned} \cos d' &= \sin\Delta \sin\delta + \cos\Delta \cos\delta \cos\alpha \\ \cos d_1 &= \sin\Delta \sin\delta_1 + \cos\Delta \cos\delta_1 \cos\alpha \end{aligned}\right\} \qquad (6)$$

from which combined with (4) we find

$$\varrho \cos d' - \varrho_1 \cos d_1 = - a i \sin\Delta$$

or by (5)

$$\cos d' - \cos d_1 = \frac{a i}{\varrho}(\sin\delta \cos d_1 - \sin\Delta)$$

$$2 \sin\tfrac{1}{2}(d' + d_1) \sin\tfrac{1}{2}(d' - d_1) = i \sin\pi\,(\sin\Delta - \sin\delta \cos d_1)$$

and with great accuracy for our present purpose,

$$d' - d_1 = \frac{i\,\pi \sin\Delta}{\sin d_1} - \frac{i\,\pi \sin\delta}{\tan d_1} \qquad (\mathbf{D})$$

a formula easily put into tables, especially if we employ a mean value of π which will never produce an error of more than about 1″. If any one, however, desires to compute this correction directly, it may be done by the formula

$$d' - d_1 = N\pi \sin\varphi \,.\, \frac{\sin\Delta}{\sin d_1} - N\pi \sin\varphi \,.\, \frac{\sin\delta}{\tan d_1} \qquad (\mathbf{D^*})$$

in which

$$N = \frac{\epsilon^2}{\sqrt{(1-\epsilon^2 \sin^2\varphi)}}$$

and we may employ without sensible error the value of N corresponding to $\varphi = 45°$, or $\log N = 7.8170$, the compression being $\frac{1}{300}$.

The computation of this correction would be rendered at once simple and accurate in practice, if the ephemeris contained the log of

$$N\pi\left(\frac{\sin\Delta}{\sin d'} - \frac{\sin\delta}{\tan d'}\right) = N'$$

(which is equivalent to a logarithm introduced by Bessel into his ephemeris for the same purpose), for we should then have

$$d' - d_1 = N' \sin\varphi. \qquad (7)$$

IV.

Corrections for the Contraction of the Moon's and Sun's Semidiameters by Refraction. — The apparent distance of the centers of the moon and sun has been supposed above to have been found in the usual manner from the observed distance of the limbs, by adding the apparent semidiameters; or when the moon has been observed with a planet or star, by adding or subtracting the moon's semidiameter alone, according as her nearest or farthest limb has been observed. At low altitudes the elliptical figure of the disc must be taken into consideration; for the refraction being different at points of the limb which have different altitudes, the result is an apparent contraction of every semidiameter, the vertical ones being the most, and those perpendicular to the vertical the least contracted.* It becomes necessary to obtain a general expression for the contraction of that semidiameter which lies in the direction of the distance, and makes an angle q with the vertical circle. If we put

s = horizontal semidiameter of the moon + the augmentation,
s_0 = the apparent vertical semidiameter,
s' = " inclined "
Δs_0 = contraction of vertical " $= s - s_0$
$\Delta s'$ = " inclined " $= s - s'$
Δr = difference of refractions at the center of the moon and the observed point on the limb,

we have nearly $\Delta s' = \Delta r \cos q$.

But the apparent altitude of the center being h, Δs_0 is the difference of refractions at the apparent altitudes h and $h + s_0$, while Δr is the difference of refractions at h and $h + s' \cos q$, whence

$$\Delta s_0 : \Delta r = s_0 : s' \cos q$$

$$\Delta r = \frac{s'}{s_0} \,.\, \Delta s_0 \cos q = \Delta s_0 \cos q \text{ (nearly)}$$

$$\Delta s' = \Delta s_0 \cos^2 q \qquad (8)$$

a known formula, which agrees very nearly with the hypothesis that the figure of the disc is an ellipse. It is evident, however, that the lower half of the disc is more flattened than the upper half; but if Δs_0 be taken as the *mean* of the contrac-

* It is usually stated that the semidiameter which is perpendicular to the vertical circle is *not* contracted; but in fact it is contracted by a small quantity, which is nearly the same at all altitudes; for if Z denote the azimuthal angle between the center and the extremity of a horizontal semidiameter, s the true and s' the apparent horizontal semidiameter, we have

$$\tan Z = \frac{\tan s}{\cos(h - r)} = \frac{\tan s'}{\cos h}$$

whence we easily deduce

$$s - s' = s' r \sin 1'' \tan h = s' k \sin 1''$$

in which $k = r \tan h$ is nearly constant. If $s = 16'$ we have for $h = 5°$, $s - s' = 0''.24$; and for $h = 90°$, $s - s' = 0''.27$. This small quantity is not taken into account in the investigation in the text.

tions of the upper and lower vertical semidiameters, the preceding formula will be in error only 0″.4 at the altitude 10°, and 1″.2 at 5°; the maximum values of $\varDelta s_0$ at those altitudes being respectively 10″ and 30″. The changes of the thermometer and barometer may also sensibly affect the value of $\varDelta s_0$ at low altitudes, but only by 4″ in the improbable case of the *highest barometer* and *lowest thermometer*, and $h = 5°$. It will hardly be necessary to attend to this small error in practice; nevertheless, it can readily be done without any further reference to the refraction tables, for the computer will already have before him r_0' the mean value of r', and $\varDelta r_0'$ the sum of the corrections of r_0' for barometer and thermometer; so that he may find at once the proportional correction of $\varDelta s'$, which is $\varDelta s' . \frac{\Delta r'}{r_0'}$.

Now the angle q is given by the formula

$$\cos q = \frac{\sin H - \sin h \cos d}{\cos h \sin d}$$

and we have from the formulas (**A**)

$$\frac{B'}{B(p-r')\cos h} = -\frac{\sin H}{\cos h \sin d}, \quad \frac{A'}{A\,(p-r')\cos h} = \frac{\sin h \cos d}{\cos h \sin d},$$

so that

$$\cos q = -\left(\frac{B'}{B} + \frac{A'}{A}\right)\frac{1}{(p-r')\cos h}.$$

If we assume $A = 1$, $B = 1$, we shall have

$$\cos q = -\frac{A' + B'}{(p-r')\cos h}$$

$$\varDelta s' = \varDelta s_0 . \frac{(A'+B')^2}{(p-r')^2 \cos^2 h} \qquad \textbf{(E)}$$

which is easily put into tables. A table with the arguments h and $p - r'$ may give the value of

$$\frac{\Delta s_0}{(p-r')^2 \cos^2 h}$$

and a second table with the arguments $A' + B'$ and "the number from the first table" may give $\varDelta s'$.

In order to ascertain the degree of accuracy of the formula (**E**), we observe that the errors in $\cos q$ produced by taking $A = 1$, $B = 1$, are

$$e = (A-1)\frac{\tan h}{\tan d}, \qquad e' = (1-B)\frac{\sin H}{\cos h \sin d};$$

the errors in $\cos^2 q$ are

$$2\,e \cos q \qquad \text{and} \qquad 2\,e' \cos q$$

and the errors in $\varDelta s'$ are therefore

$$e_1 = \frac{2\,\Delta s_0\,(A-1)\tan h \cos q}{\tan d}, \quad e_1' = \frac{2\,\Delta s_0\,(1-B)\sin H \cos q}{\cos h \sin d}.$$

The greatest values of e_1 and e_1' at different altitudes are shown as follows, taking $\cos q = 0$, $H = 90°$, in order to represent the extreme cases:—

h	$e_1 \tan d$	$e_1' \sin d$
5°	0″.45	0″.02
10°	0″.16	0″.00
15	.08	.00
30	.02	.00
50	.00	.00

It appears, therefore, that the error of the formula (**E**), like that of (8), becomes sensible only at those low altitudes where extreme precision is unattainable on account of the uncertainty of the refraction. We may therefore safely employ it as sufficiently accurate for all cases.

When the sun is observed with the moon, a similar correction must be applied to his semidiameter. If

Q = angle at the sun,
S = true semidiameter of the sun,
S_0 = apparent vertical semidiameter of the sun,
S' = " inclined " "
$\varDelta S_0$ = contraction of vertical semidiameter $= S - S_0$,
$\varDelta S'$ = " inclined " $= S - S'$,

then as above

$$\varDelta S' = \varDelta S_0 \cos^2 Q$$

$$\cos Q = \frac{\sin h - \sin H \cos d}{\cos H \sin d} = \left(\frac{C'}{C} + \frac{D'}{D}\right)\frac{1}{(R'-P)\cos H}$$

and assuming $C = 1$, $D = 1$, we have

$$\cos Q = \frac{C' + D'}{(R'-P)\cos H}$$

$$\varDelta S' = \varDelta S_0 . \frac{(C'+D')^2}{(R'-P)^2 \cos^2 H} \qquad \textbf{(F)}$$

which is even more accurate than (**E**), and is put into tables in the same manner.

The corrections $\varDelta s'$ and $\varDelta S'$ should strictly be applied to the semidiameters, and should appear in the value of d employed in the computation of $\varDelta d$; but since the values of A', B', C', and D' are required in finding $\varDelta s'$ and $\varDelta S'$, we have to employ a value of d which may in extreme cases be in error by about 30″. This produces a small error in each of the terms A', B', C', D', which could in practice be eliminated only by repeating the computation with the corrected value of d. But this repetition is unnecessary, as the error in $\varDelta d$ is rarely more than 0″.5; and it will suffice to apply $\varDelta s'$ and $\varDelta S'$ directly to d_1.

In order, however, to show generally the effect upon $\varDelta d$ of small errors in d, let us differentiate the equation (**C**), regarding the term x (of the second order) as constant, and taking $A = 1$, $B = 1$, $C = 1$, $D = 1$ (which also amounts to considering terms of the second order as constant). We find

$$\delta \varDelta d = -\frac{(p-r')(\sin h - \sin H \cos d)\sin 1''\,\delta d}{\sin^2 d} + \frac{(R'-P)(\sin H - \sin h \cos d)\sin 1''\,\delta d}{\sin^2 d}$$

or

$$\delta \varDelta d = -[(p-r')\cos Q \cos H - (R'-P)\cos q \cos h]\frac{\sin 1''\,\delta d}{\sin d}. \qquad (9)$$

This formula shows at once that the maximum of $\delta \varDelta d$ occurs when the two bodies are in the same vertical circle, the moon being the higher body; for this condition gives $\cos q = -1$, $\cos Q = 1$, so that the two terms obtain the same sign, and at the same time $p - r'$ and $R' - P$ have their greatest values. In this position, we have $d = h - H$, and the formula for the maximum of $\delta \varDelta d$ is therefore

$$\delta_0 \varDelta d = -[(p - r') \cos H + (R' - P) \cos h] \frac{\sin 1'' \delta d}{\sin (h - H)}. \qquad (10)$$

The following table shows the maximum effect upon $\varDelta d$ of an error of 1′ in d, computed by formula (10), for the several values of h and H; the least value of $h - H$ (= d) being 20°.

H	h 25°	35°	45°	55°	65°	75°	85°	90°
°	″	″	″	″	″	″	″	″
5	3.6	2.4	1.9	1.5	1.3	1.2	1.1	1.1
15		3.2	2.2	1.7	1.4	1.2	1.1	1.0
25			2.9	2.0	1.5	1.3	1.1	1.0
35				2.6	1.8	1.4	1.1	1.0
45					2.2	1.5	1.2	1.0
55						1.8	1.2	1.0
65							1.3	0.9

This table of *extreme* errors shows clearly enough that the error arising from the neglect of $\varDelta s'$ and $\varDelta S'$ in the value of d employed in computing $\varDelta d$, is too small to require any departure from the process already indicated. For the navigator must bear in mind that all observations at very low altitudes are subject to two principal sources of error; — 1st, the uncertainty of the refraction, which no process of calculation can eliminate; and 2d, the imperfect definition of the limb of the moon or sun in the vicinity of the horizon. If a method of computation involves only errors which in every case are less than these unavoidable errors, it satisfies the essential condition of a good method.

FROM A LETTER OF PROFESSOR SAWITSCH, DIRECTOR OF THE ST. PETERSBURG OBSERVATORY, TO THE EDITOR.

St. Petersburg, 1851, *May* 27.

Accompanying this are some observations which I send for insertion in your *Astronomical Journal*. They give the apparent positions of the centers, at the time of their transit over the St. Petersburg meridian.

OPPOSITION OF SATURN, 1849.

Date. 1849.	♄ α h. m. s.	Obs. — Naut. Alm. s.	♄ δ	Obs. — Naut. Alm.	Comparison-Stars.
Sept. 24	0 21 44.23	+1.24	−0° 31′ 30″.4	+40″.5	33 *Piscium*
26	21 10.33	1.68	35 18.7	38.9	and
Oct. 3	19 10.18	1.26	48 23.2	36.5	20 *Ceti.*
6	0 18 19.62	+1.60	−0 53 47.6	+38.5	

The observed declinations are freed from the effects of parallax and refraction.

OPPOSITION OF URANUS, 1849.

Date. 1849.	♅ α h. m. s.	Obs. — Naut. Alm. s.	♅ δ	Obs. — Naut. Alm.	Comparison-Stars.
Oct. 3	1 32 55.63	−10.33	+9° 4′ 3″.6	+16″.8	*μ Piscium* and
15	1 31 7.07	−10.18	+8 53 30.8	+11.1	*ν Piscium.*

The observed declinations are freed from the effects of parallax and refraction.

MARS, 1849.

Date. 1849.	♂ α h. m. s.	Obs. — Naut. Alm. s.	♂ δ	Comparison-Stars.
Dec. 21	5 37 19.89	+1.85	+26° 30′ 14″.0	125 *Tauri.*
23	33 58.94	1.95	31 0.1	*β* and 118 *Tauri.*
26	5 29 9.98	+1.56	+26 29 8.4	*β* and 118 *Tauri.*

The observed declinations are corrected for refraction only; no regard having been had to the effect of parallax.