Properties of Confidence Ellipses

The probability density function for the observer's position given in Equation (12) is written in its properly normalized form as

$$\mathrm{pr}(x,y)=\frac{\sqrt{4ac-b^2}}{2\pi}\exp\left\{-\left(ax^2+bxy+cy^2+dx+ey+f-p\right)\right\} \tag{C1}$$

where

$$a=\frac{1}{2}\sum_i\frac{1}{\sigma_i^2}\sin^2 Z_i;\qquad b=\sum_i\frac{1}{\sigma_i^2}\sin Z_i\cos Z_i;\qquad c=\frac{1}{2}\sum_i\frac{1}{\sigma_i^2}\cos^2 Z_i$$

$$d=-\sum_i\frac{1}{\sigma_i^2}r_i\sin Z_i;\qquad e=-\sum_i\frac{1}{\sigma_i^2}r_i\cos Z_i;\qquad f=\frac{1}{2}\sum_i\frac{1}{\sigma_i^2}r_i^2$$

and p is defined in Appendix A.

The quadratic argument of the exponential function defines a set of confidence ellipses for the observer's position like those shown in Figure 2. The properties of these confidence ellipses can provide a quantitative measure as to the relative quality of a round of sights. For example the area of the ellipse at a specified confidence level or the root-mean-square deviations of possible positions from the centre might serve as suitable measures.

Methods for determining the centre, orientation and lengths of the semi-major and semi-minor axis lengths of an ellipse from a quadratic form are well established and described elsewhere (Pettofrezzo 1978). Applying these methods and performing the appropriate algebraic reduction yields relatively simple and efficient expressions for these properties in terms of the intercepts, r_i, and azimuths, Z_i, of an arbitrary number of LoP's.

If the probability of the observer's position falling inside a given confidence ellipse is P then it is known (Abramowitz and Stegun 1964 26.3.21, Daniels 1951) that the ellipse satisfies the condition

$$ax^2+bxy+cy^2+dx+ey+f-p=L \tag{C2}$$

where $L=-\ln(1-P)$. To determine the characteristics of the ellipse it is convenient to start by writing Equation (C2) in the form

$$\vec{x}\cdot A_Q\cdot\vec{x}^{\mathrm{T}}\equiv\begin{pmatrix}x & y & 1\end{pmatrix}\begin{pmatrix}a & b/2 & d/2\\ b/2 & c & e/2\\ d/2 & e/2 & f-p-L\end{pmatrix}\begin{pmatrix}x\\ y\\ 1\end{pmatrix}=0.$$

The matrix $A_{33}=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}$, whose determinant is the minor or cofactor of the bottom right hand element in matrix A_Q, is also needed. Its trace and determinant are

$$\mathrm{Tr}\,A_{33}=\frac{1}{2}\sum_i\frac{1}{\sigma_i^2};\qquad \left|A_{33}\right|=\frac{1}{8}\sum_{i,j}\frac{1}{\sigma_i^2\sigma_j^2}\sin^2\left(Z_i-Z_j\right).$$

For the particular matrix elements encountered here $\left|A_Q\right| = -L\left|A_{33}\right|$ which leads to significant simplifications in the results that follow.

The centre of the ellipse is the MPP which lies at

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{4\left|A_{33}\right|} \begin{pmatrix} \sum_{i,j} \frac{1}{\sigma_i^2 \sigma_j^2} r_i \cos Z_j \sin\left(Z_i - Z_j\right) \\ -\sum_{i,j} \frac{1}{\sigma_i^2 \sigma_j^2} r_i \sin Z_j \sin\left(Z_i - Z_j\right) \end{pmatrix}.$$

The principal axes of the ellipse are the eigenvectors of A_{33} and are found to be

$$\left(\frac{1}{2} \sum_i \frac{1}{\sigma_i^2} \sin 2Z_i, \frac{1}{2} \sum_i \frac{1}{\sigma_i^2} \cos 2Z_i \pm \sqrt{\left(\mathrm{Tr}\, A_{33}\right)^2 - 4\left|A_{33}\right|} \right)$$

or equivalently

$$\left(-\frac{1}{2} \sum_i \frac{1}{\sigma_i^2} \cos 2Z_i \pm \sqrt{\left(\mathrm{Tr}\, A_{33}\right)^2 - 4\left|A_{33}\right|}, \frac{1}{2} \sum_i \frac{1}{\sigma_i^2} \sin 2Z_i \right)$$

which have associated eigenvalues

$$\lambda_\pm = \frac{1}{2} \mathrm{Tr}\, A_{33} \pm \frac{1}{2} \sqrt{\left(\mathrm{Tr}\, A_{33}\right)^2 - 4\left|A_{33}\right|}.$$

From this it follows that the semi-major axis, $a = \sqrt{L/\lambda_-}$, and semi-minor axes, $b = \sqrt{L/\lambda_+}$. The area of this ellipse is then $\pi ab = \pi L \Big/ \sqrt{\left|A_{33}\right|}$.

The root-mean-square distance of possible positions from the MPP is $\sqrt{\tfrac{1}{2} \mathrm{Tr}\, A_{33} \Big/ \left|A_{33}\right|}$.

Stansfield (1947) treats the problem of the probability distribution for a fix generated by an arbitrary number of radio direction finding stations. The problem is transformed into one that is analytically equivalent to the one treated in this appendix. Expressions for the properties of the confidence ellipses are given that are equivalent to those above but without the additional trigonometric reduction that has been performed here.