Daylength
Lloyd Sakazaki
May 25, 2016

> *Sunrise, sunset,*
> *Sunrise, sunset,*
> *Swiftly fly the years,*
> *One season following another,*
> *Laden with happiness and tears.*
>
> *What words of wisdom can I give them . . .?*
>
> (Lyrics from "Sunrise, Sunset," *Fiddler on the Roof*, 1964)

The Sun rises and sets through the seasons. Days are shortest in the winter and longest in the summer. How might we calculate the length of a day—any day—of the year here on planet Earth?

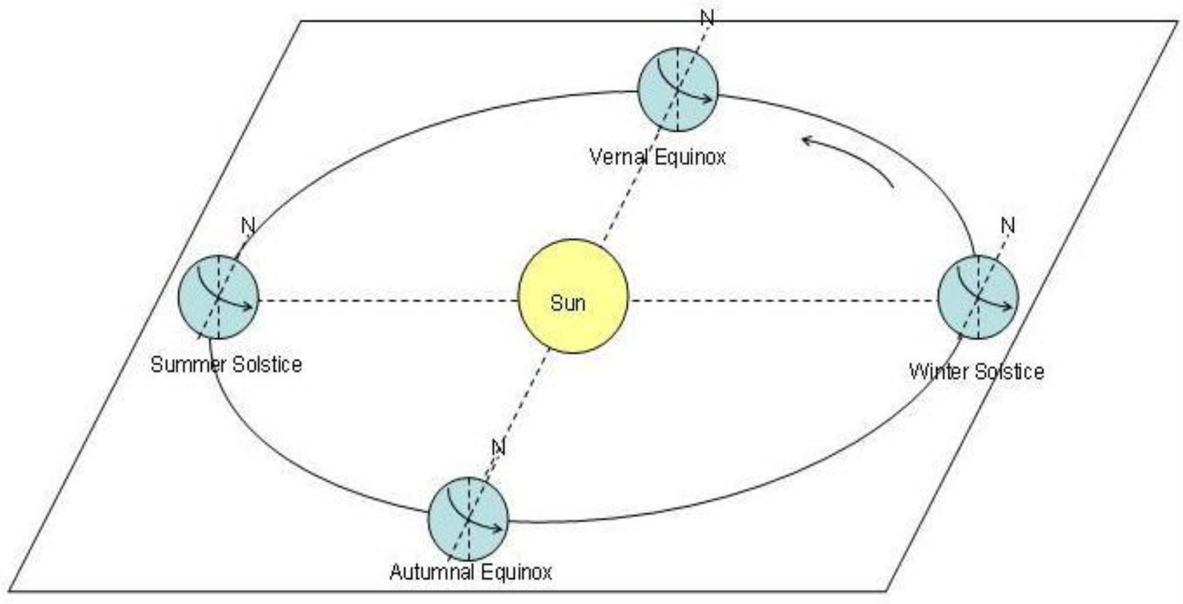
In this write-up, I provide two methodologies for calculating daylength:

- **Rectangular Coordinate-Based Analysis**: This calculation relies on little more than logical thinking and high school-level algebra and (plane) trigonometry. The analysis is "simple" but the algebra is somewhat cumbersome, requiring a modicum of patience to work through the details.
- **Spherical Trigonometry "Coordinate-Free" Formulation**: This methodology relies on a theorem from spherical trigonometry, specifying how the parts of a triangle lying on the surface of a sphere relate to one another. The analysis adopts the geometrical framework used by astronomers and readily produces the desired solution.

My objective is to display the power of both simple and more sophisticated analysis. The former provides an example of what can be accomplished by diligently employing only basic mathematical techniques, while the latter exhibits a type of coordinate-free "elegance" best appreciated after having slogged through the lengthier coordinate-based analysis.

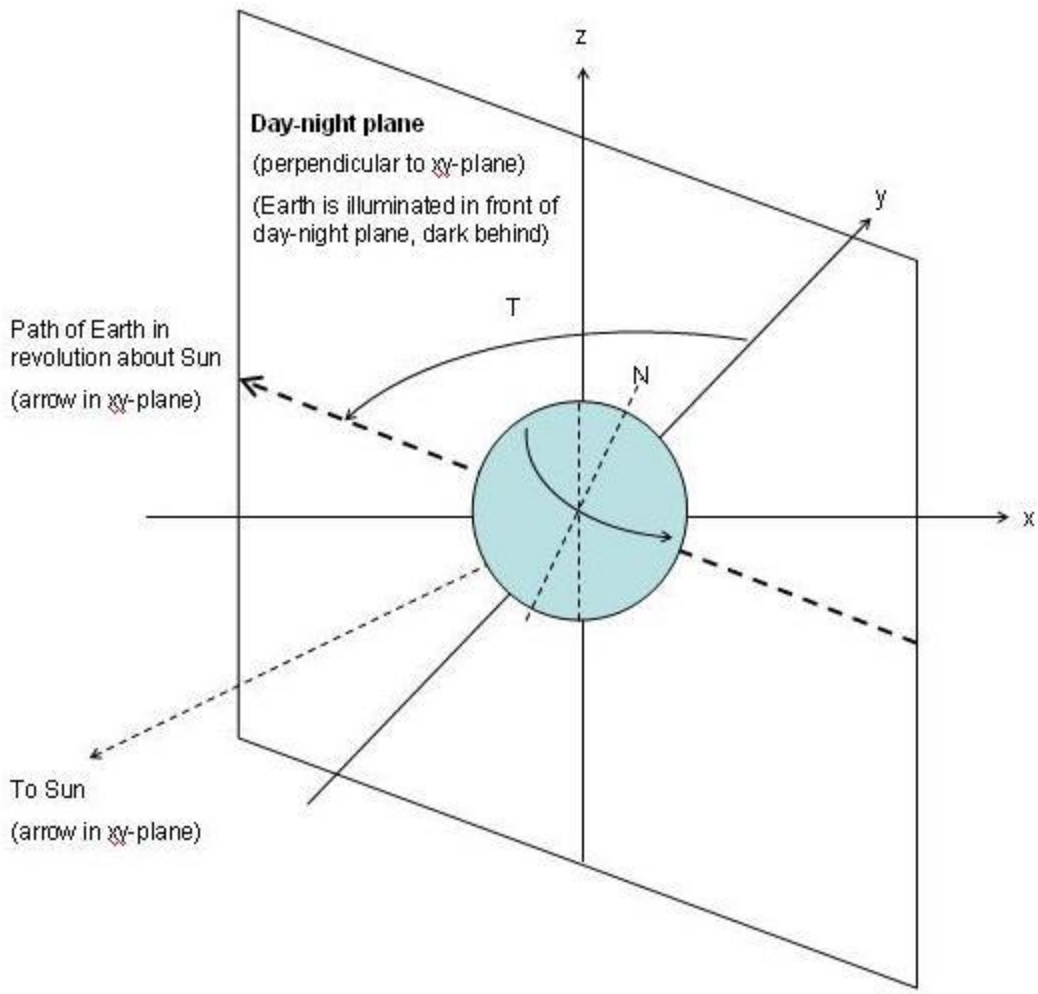
Earth-Sun System in Rectangular Coordinates

Consider the Earth-Sun plane within which both the Earth and the Sun reside. The diagram below shows the solar-centric Copernican viewpoint, with the Earth revolving about the Sun, passing through its four seasons demarcated for northern hemisphere by the winter solstice (approximately December 21), vernal equinox (March 21), summer solstice (June 21) and autumnal equinox (September 21). The seasons, of course, are (by north-south symmetry) reversed for southern hemisphere locations.



Instead of following Copernicus by taking the Sun to be stationary, we can alternatively take an Earth-centric viewpoint with the Sun revolving about the Earth. Define rectangular coordinates with the Earth and Sun in the horizontal xy-plane, such that the positive x- and y-axes point away from the Sun in the direction of the winter solstice and vernal equinox, respectively, and the positive z-axis points vertically upward perpendicular to the Earth-Sun plane. In this set-up, the Earth's tilt (about 23.5º) is the angle between the z-axis and the direction of polar north of the Earth, conceptually equivalent to a one-time 23.5º rotation of the Earth (centered at the origin) about the y-axis.

Now, day and night occur because the Earth rotates once every 24 hours about its north-south polar axis. Since the Sun illuminates the "front" hemisphere of the Earth's surface facing the Sun at any given moment, while the Earth's "back" hemisphere is dark, consider the day-night plane oriented perpendicular to the Sun's rays reaching the Earth's surface. At the winter solstice, this day-night plane is the yz-plane ($x = 0$). As days pass in the calendar year, this plane gradually rotates about the z-axis, so that at the time of the vernal equinox it becomes aligned with the xz-plane ($y = 0$). The diagram below shows the position of the day-night plane on a calendar day occurring between the winter solstice and vernal equinox.



To keep track of time during the calendar year, define T as an angle in the xy-plane representing the amount of rotation of the day-night plane away from its winter solstice position, so that T = 0º (plane x = 0) at the winter solstice, increasing to T = 90º (plane y = 0) at the vernal equinox. Generally, the equation of the day-night plane is

$$\tan T = -x/y. \qquad \text{(Eqn. 1)}$$

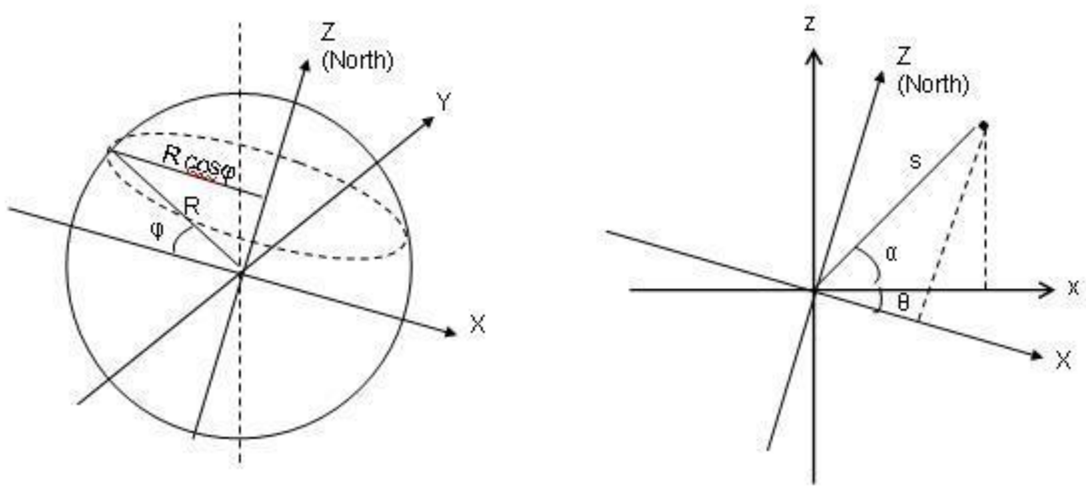
Note that z does not appear in this equation because the day-night plane is always perpendicular to the xy-plane. This equation effectively parameterizes the rotation of the day-night plane as a function of T between 0º and 360º.during the full calendar year.

Next, because daylength is determined by the intersection of circles of constant latitude on the Earth's surface with the day-night plane, we need to find the equations representing these circles. (To be precise, between sunrise and sunset on any given day, the day-night plane rotates slightly, which impacts daylength by up to one part in 365, or about 0.3%. For the purposes of the present calculation, we will consider this to be a secondary effect small enough to ignore.)

In a "tilted" reference frame (denoted by upper case rectangular coordinates X, Y and Z) with the Z-axis aligned in the direction of polar north, circles of constant latitude are given by the familiar equation for a circle, situated at the appropriate height Z above or below the XY-plane:

$$X^2 + Y^2 = r^2 = (R \cos \varphi)^2, \;\; -90^\circ \le \varphi \le 90^\circ, \qquad \text{(Eqn. 2a)}$$
$$Z = R \sin \varphi, \qquad \text{(Eqn. 2b)}$$

where r is the radius of the circle, R is the radius of the Earth (assumed to be a perfect sphere), and φ is the latitude (ranging from -90º at the South Pole to 90º at the North Pole). An example of such a circle is shown at the left of the diagram below.



In order to return to our original "untilted" reference frame (denoted by lower case x, y and z), we need to rotate the X- and Z-axes clockwise about the Y-axis through the tilt angle θ. With a general point in the untilted reference frame represented in polar coordinates as (x, z) = (s cos α, s sin α), the corresponding values of the tilted (X, Z) coordinates (see right side of above diagram) are

$$X = s \cos(\alpha + \theta) = s \cos \alpha \cos \theta - s \sin \alpha \sin \theta = x \cos \theta - z \sin \theta$$
$$Z = s \sin(\alpha + \theta) = s \sin \alpha \cos \theta + s \cos \alpha \sin \theta = x \sin \theta + z \cos \theta. \qquad \text{(Eqn. 3)}$$

Therefore, by substituting these expressions for X and Z into Eqns. 2a and 2b, and using Y = y (since y values are invariant in the rotation), we arrive at

$$(x \cos \theta - z \sin \theta)^2 + y^2 = r^2 = (R \cos \varphi)^2, \;\; -90^\circ \le \varphi \le 90^\circ, \qquad \text{(Eqn. 4a)}$$
$$x \sin \theta + z \cos \theta = R \sin \varphi \qquad \text{(Eqn. 4b)}$$

for the *tilted* circles of constant latitude in our original untilted (x, y, z) reference frame.

Intersection of Day-Night Plane and Tilted Circles of Constant Latitude

To find the intersection of the tilted circles of constant latitude with the day-night plane, we simultaneously solve Eqn. 1 and Eqns. 4a and 4b. By Eqn. 1,

$$y = -x \cot T. \qquad \text{(Eqn. 5)}$$

Also, by Eqn. 4b,

$$z = (-x \sin \theta + R \sin \varphi)/\cos \theta. \qquad \text{(Eqn. 6)}$$

Substitution of these expressions for y and z into Eqn. 4a gives

$$[x \cos \theta - (-x \sin \theta + R \sin \varphi) \tan \theta]^2 + x^2 \cot^2 T = R^2 \cos^2\varphi.$$

Collecting terms by like powers of x and simplifying, we have

$$[(\cos \theta + \sin^2\theta/\cos \theta)\, x - R \tan \theta \sin \varphi]^2 + \cot^2 T\, x^2 = R^2 \cos^2\varphi$$
$$[\sec^2\theta + \cot^2 T]\, x^2 - (2R \sec \theta \tan \theta \sin \varphi)\, x + R^2 (\tan^2\theta \sin^2\varphi - \cos^2\varphi) = 0$$
$$[\sec^2\theta + \cot^2 T]\, x^2 - (2R \sec^2\theta \sin \theta \sin \varphi)\, x + R^2 (\sec^2\theta \sin^2\varphi - 1) = 0$$
$$[1 + \cos^2\theta \cot^2 T]\, x^2 - (2R \sin \theta \sin \varphi)\, x + R^2 (\sin^2\varphi - \cos^2\theta) = 0.$$

Defining the dimensionless variable

$$u = x/R, \qquad \text{(Eqn. 7)}$$

we can rewrite the prior equation as

$$(1 + \cos^2\theta \cot^2 T)\, u^2 - (2 \sin \theta \sin \varphi)\, u + (\sin^2\varphi - \cos^2\theta) = 0,$$

which is quadratic in u and has solutions

$$u_{1,2} = \{\sin \theta \sin \varphi \pm [\sin^2\theta \sin^2\varphi - (1 + \cos^2\theta \cot^2 T)(\sin^2\varphi - \cos^2\theta)]^{1/2}\}/(1 + \cos^2\theta \cot^2 T). \qquad \text{(Eqn. 8)}$$

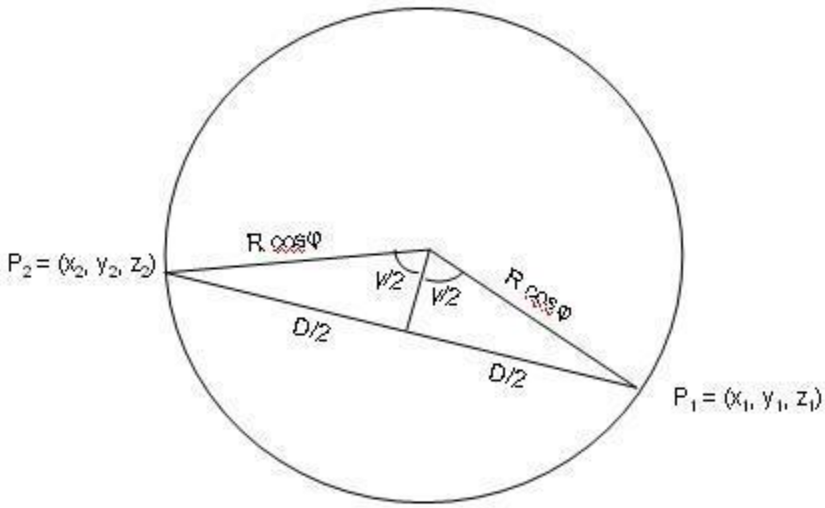
We can also rewrite y and z from Eqns. 5 and 6 in terms of the dimensionless variables v and w, respectively,

$$v = y/R = -u \cot T, \qquad \text{(Eqn. 9)}$$
$$w = z/R = (-u \sin \theta + \sin \varphi)/\cos \theta, \qquad \text{(Eqn. 10)}$$

which are both now expressed in terms of u.

Daylength

Given that we have now determined the two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, where any particular circle of constant latitude intersects the day-night plane, we can proceed to calculate daylength.



The diagram above shows a circle of constant latitude viewed from polar north. The (linear) distance between P_1 and P_2 is

$$D = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2},$$

or in terms of u, v and w,

$$d = D/R = [(u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2]^{1/2}. \qquad \text{(Eqn. 11)}$$

As seen in the diagram, the angular distance between P_1 and P_2 can be determined by the relation

$$\sin(\gamma/2) = (D/2)/(R \cos \varphi) = d/(2 \cos \varphi). \qquad \text{(Eqn. 12)}$$

Therefore, based on the convention of a 24-hour day, daylength becomes

$$\begin{aligned} \text{daylength} &= \gamma\ (24 \text{ hours}/360^\circ) \\ &= 2 \arcsin(d/(2 \cos \varphi))\ (24 \text{ hours}/360^\circ). \qquad \text{(Eqn. 13)} \end{aligned}$$

with γ (hence also the value of the arcsin function) taken to be in degrees.

This result can be expressed in terms of the observables θ, φ and T through an exercise in algebra. Using Eqns. 8, 9 and 10 in Eqn. 11, we have

$$d^2 = (1 + \cot^2 T + \tan^2\theta)\ (u_2 - u_1)^2$$

$= (1 + \cot^2 T + \tan^2\theta)$
$\quad \bullet\ 4 [\sin^2\theta \sin^2\varphi – (1 + \cos^2\theta \cot^2 T)(\sin^2\varphi - \cos^2\theta)]/(1 + \cos^2\theta \cot^2 T)^2$
$= (1 + \cos^2\theta \cot^2 T)/\cos^2\theta$
$\quad \bullet\ 4 [(1 - \cos^2\theta) \sin^2\varphi – (1 + \cos^2\theta \cot^2 T)(\sin^2\varphi - \cos^2\theta)]/(1 + \cos^2\theta \cot^2 T)^2$
$= 4 [(1 - \sin^2\varphi) – \cot^2 T (\sin^2\varphi - \cos^2\theta)]/(1 + \cos^2\theta \cot^2 T)$
$= 4 \cos^2\varphi [1 + \cot^2 T (1- \sec^2\varphi (1 - \cos^2\theta))]/(1 + \cos^2\theta \cot^2 T)$
$= 4 \cos^2\varphi [1 + \cot^2 T (1- (1 + \tan^2\varphi) \sin^2\theta)]/(1 + \cos^2\theta \cot^2 T)$
$= 4 \cos^2\varphi [1 - \tan^2\varphi \sin^2\theta \cot^2 T/(1 + \cos^2\theta \cot^2 T)]$
$= 4 \cos^2\varphi [1 - \tan^2\varphi \sin^2\theta \cos^2 T/(1 - \sin^2\theta \cos^2 T)]$. (Eqn. 14)

By rewriting Eqn. 12 in the form

$$\sin^2(\gamma/2) = 1 – \cos^2(\gamma/2) = d^2/(4 \cos^2\varphi),$$

we can read off from Eqn. 14 the relationship

$$\cos(\gamma/2) = \pm \tan \varphi \sin \theta \cos T/(1 - \sin^2\theta \cos^2 T)^{1/2}$$
$$= -\tan \varphi \tan \delta, \qquad \text{(Eqn. 15)}$$

where we define the parameter δ through

$$\sin \delta = \pm \sin \theta \cos T. \qquad \text{(Eqn. 16)}$$

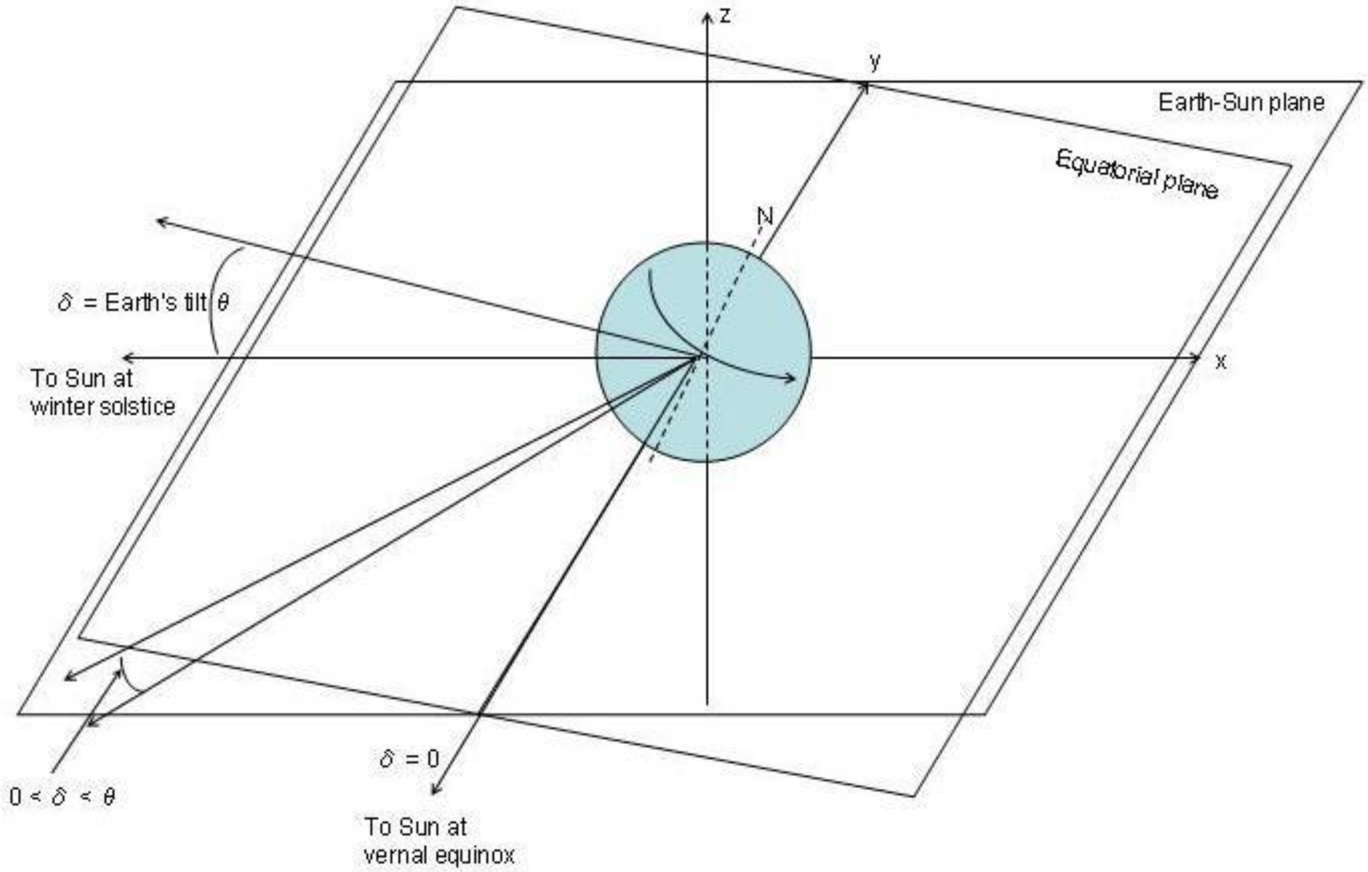
Therefore, using Eqn. 15, we may rewrite daylength in Eqn. 13 as

$$\text{daylength} = \gamma\ (24 \text{ hours}/360^o)$$
$$= 2 \arccos(-\tan \varphi \tan \delta)\ (24 \text{ hours}/360^o). \qquad \text{(Eqn. 17)}$$

Note that daylength is now expressed entirely in terms of φ and δ, with δ related to θ and T through Eqn. 16.

Understanding Declination

We can understand the physical meaning of the parameter δ by referring to the diagram below displaying the relationship between the equatorial plane (tilted with respect to the Earth-Sun plane) and the line in the Earth-Sun plane from the center of the Earth directly to the Sun.



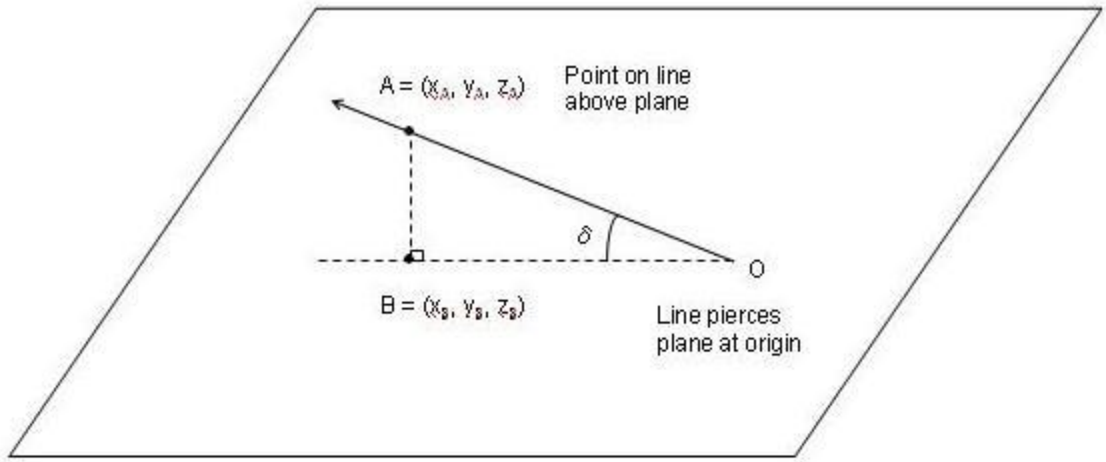
Based on our discussion of the tilted coordinates (X, Y, Z), the equation for the equatorial plane (Z = 0) is determined by setting Z = 0 in Eqn. 3 to arrive at

$$z = (-\tan\theta)\, x. \qquad \text{(Eqn. 18)}$$

Secondly, the line in the xy-plane (with z = 0) pointing from the center of the Earth to the Sun is perpendicular to the day-night plane given by y = (-1/tan T) x in Eqn. 1, and therefore has slope tan T (which is the negative multiplicative inverse of -1/tan T) and equation

$$y = (\tan T)\, x. \qquad \text{(Eqn. 19)}$$

The angle between the equatorial plane and the line to the Sun is called the declination of the Sun. One way to determine this angle is to pick a point A on the line (other than the origin) and then to drop a line segment (perpendicular to the plane) to a point B in the plane, which can be accomplished by minimizing the distance from A to B. The resulting angle AOB will be the declination of the Sun. See diagram below.



Utilizing Eqns. 19 and 18, respectively, we can write the coordinates of A and B as

$A = (x_A, y_A, z_A) = (x_A, (\tan T)\, x_A, 0)$ (Eqn. 20a)
$B = (x_B, y_B, z_B) = (x_B, y_B, (-\tan\theta)\, x_B)$ (Eqn. 20b)

Then, line segment AB has length

$$f(x_B, y_B) = [(x_B - x_A)^2 + (y_B - (\tan T)\, x_A)^2 + ((-\tan\theta)\, x_B)^2]^{1/2},$$

which is a function of two variables, x_B and y_B.

To minimize f, we set the two partial derivatives to zero

$$\partial f/\partial x_B = [2\,(x_B - x_A) + (2 \tan^2\theta)\, x_B]/(2f) = 0$$
$$\partial f/\partial y_B = [2\,(y_B - (\tan T)\, x_A)]/(2f) = 0,$$

which implies

$x_B = x_A/(1 + \tan^2\theta) = (\cos^2\theta)\, x_A$ (Eqn. 21a)
$y_B = (\tan T)\, x_A.$ (Eqn. 21b)

Due to the simple geometry of a plane pierced by a line, this single extremum of f is clearly a minimum.

By combining Eqns. 20 and 21, we can write

$$A = (x_A, (\tan T)\, x_A, 0)$$
$$B = ((\cos^2\theta)\, x_A, (\tan T)\, x_A, (-\sin\theta \cos\theta)\, x_A).$$

Then,

sin(angle AOB) = (length of AB)/(length of OA)

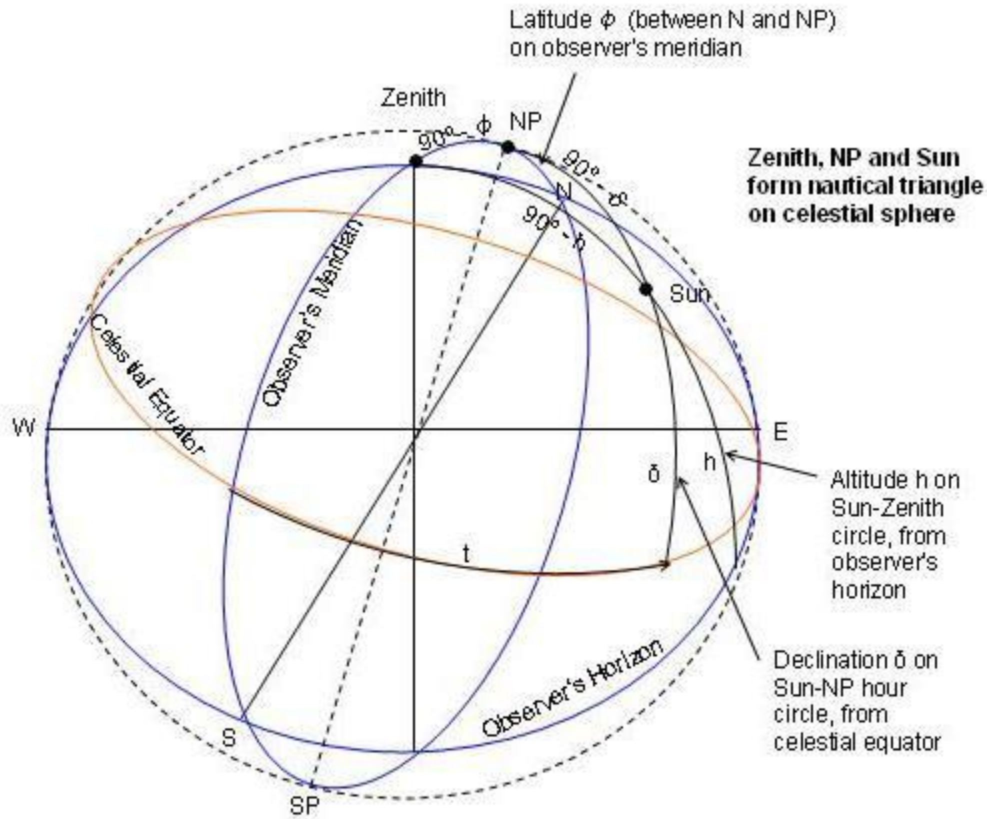
$$= [\sin^4\theta + \sin^2\theta \cos^2\theta]^{1/2}/[1 + \tan^2 T]^{1/2}$$
$$= [\sin^2\theta]^{1/2}/[\sec^2 T]^{1/2}$$
$$= \pm \sin\theta \cos T. \qquad \text{(Eqn. 22)}$$

Comparison of Eqn. 22 to Eqn. 16 reveals that our earlier parameter δ is equal to angle AOB. In other words, δ is precisely the declination of the Sun.

Coordinate-Free Formulation on the Celestial Sphere

The same problem of calculating daylength can be solved in a coordinate-free context by carefully examining relevant great circles of the celestial sphere having an observer located at its center (see diagram below):

- **Observer's Horizon**: This great circle lies in the horizontal plane of the observer, allowing for definition of the benchmark navigational directions: north (N), south (S), east (E) and west (W).
- **Observer's Meridian**: The observer's meridian is perpendicular to the observer's horizon and passes through all of the following points: observer's zenith (point directly overhead), observer's north (N), observer's south (S), the North Pole (NP) and the South Pole (SP)—all projected onto the celestial sphere.
- **Sun-Zenith Circle**: The observational position of the Sun in the sky defines the great circle passing through the Sun's projection onto the celestial sphere and the observer's zenith. This great circle is also perpendicular to the observer's horizon.
- **Celestial Equator**: Projection of the Earth's equator onto the celestial sphere gives the celestial equator, which lies in the plane perpendicular to the polar line defined by the North Pole and the South Pole.
- **Sun-North Pole Hour Circle**: The great circle passing through the projections of the Sun and the North Pole (and the South Pole) onto the celestial sphere is perpendicular to the celestial equator. This great circle is called the hour circle of the Sun.

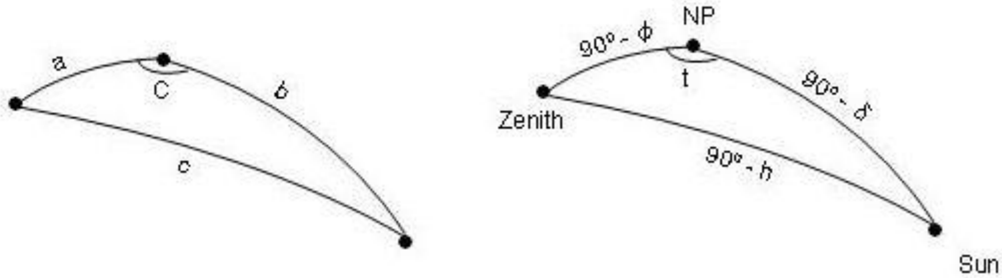


Using these great circles, we now focus attention on the nautical triangle sitting on the surface of the celestial sphere, defined by the three vertices: observer's zenith, North Pole and Sun. The sides of this triangle have lengths that can be expressed by angular measures:

- **Side from Zenith to North Pole**: The angle between projections of the observer's due north (N) and the North Pole (NP) onto the celestial sphere—being 0º when the observer is situated on the Earth's equator and reaching 90º in the limit as the observer approaches the North Pole—is equal to the observer's latitude φ. Also, the angular distance between the observer's due north and the observer's zenith is 90º. Therefore, the length of the side from zenith to NP is the difference 90º - φ.
- **Side from North Pole to Sun**: The angle between the celestial equator and the Sun is the Sun's declination δ. Also, the angle between the celestial equator and the projection of the North Pole is 90º. Therefore, the length of the side from NP to the Sun is 90º - δ.
- **Side from Sun to Zenith**: The angle between the observer's horizon and the Sun is the Sun's altitude h. Also, the angle between the observer's horizon and zenith is 90º. Therefore, the length of the side from Sun to observer's zenith is 90º - h.

As can be seen in the diagram above, the angle at the North Pole vertex NP runs along the celestial equator between the observer's meridian and the Sun-North Pole hour circle. Call this angle t.

With the necessary parts of the triangle now defined, we invoke a well-known formula from spherical trigonometry, analogous to the law of cosines in plane trigonometry. (Incidentally, proof of the spherical trigonometry result, called the "spherical law of cosines," follows directly from appropriate application of the more familiar plane trigonometry law of cosines.) The geometry is shown in the diagram below.



The spherical law of cosines states that for any triangle on the surface of a sphere, such that sides a, b and c all lie on great circles, and angle C is subtended by side c,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

As displayed in the diagram, application of the spherical law of cosines to our zenith-NP-Sun triangle gives

$$\cos (90^\circ - h) = \cos (90^\circ - \varphi) \cos (90^\circ - \delta) + \sin (90^\circ - \varphi) \sin (90^\circ - \delta) \cos t,$$

which can be re-expressed as

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos t. \quad \text{(Eqn. 23)}$$

Now, because we are interested in calculating daylength, we set $h = 0$ in Eqn. 23, corresponding to zero altitude for the Sun at both sunrise and sunset. Then, after rearranging, we have

$$\cos t = -\tan \varphi \tan \delta, \quad \text{(Eqn. 24)}$$

which is called the "sunrise equation."

Referring back to Eqn. 15, we identify t as being equal to $\gamma/2$. The physical meaning of t is that it represents the angular time (with 24 hours equivalent to 360º) from sunrise to local solar noon (when the Sun reaches its culmination point in the sky), which by symmetry also equals the time from local solar noon to sunset. In other words, daylength, which is defined as the time from sunrise to sunset, equals 2t.

Therefore, using Eqn. 24, we may write

$$\begin{aligned} \text{daylength} &= 2t\ (24\ \text{hours}/360^\circ) \\ &= 2\ \arccos(-\tan\varphi\ \tan\delta)\ (24\ \text{hours}/360^\circ), \qquad \text{(Eqn. 25)} \end{aligned}$$

which is the same as Eqn. 17.

Arguably, this geometrical coordinate-free analysis is more elegant than the algebra-intensive coordinate-based approach pursued earlier. However, in my opinion, the advantage of adopting the more sophisticated geometry and spherical trigonometry on the celestial sphere must be weighed against an incumbent trade-off—namely, a masking of the solar-centric Copernican framework with its intrinsically simpler "flat" geometry of lines, planes, and circles (although tilted they may be!).

Example: Calculated vs. Actual Daylength

Recall from Eqn. 15 that

$$\tan\delta = \pm\ \sin\theta\ \cos T/(1 - \sin^2\theta\ \cos^2 T)^{1/2}.$$

Therefore, we may write daylength in Eqn. 17 or Eqn. 25 as

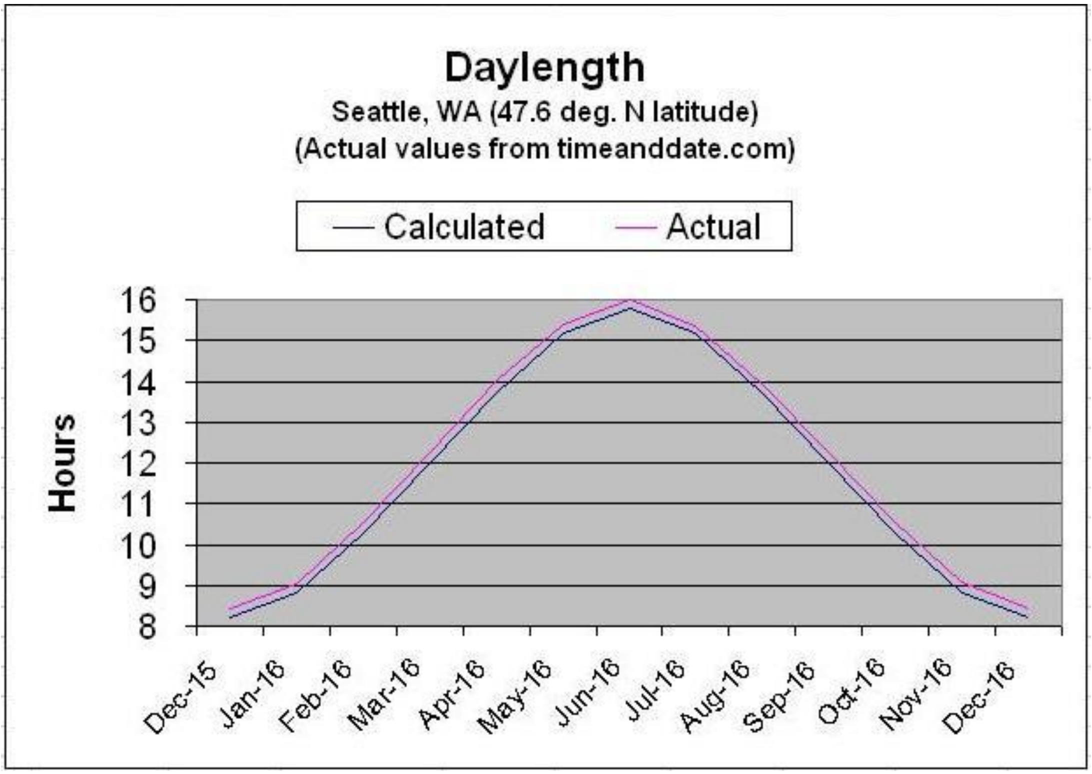
$$\text{daylength} = 2\ \arccos(\sin\theta\ \tan\varphi\ \cos T/(1 - \sin^2\theta\ \cos^2 T)^{1/2})\ (24\ \text{hours}/360^\circ),$$

.

choosing the positive sign inside of the argument of the arccos function, to match the proper relationship between season or time of year (T), the Sun's declination (δ) and daylength for northern hemisphere locations:

$T = 0^\circ$ (winter solstice) → 90° (vernal equinox), $\delta < 0$, daylength = 0 → 12 hrs.
$T = 90^\circ$ (vernal equinox) → 180° (summer solstice), $\delta > 0$, daylength = 12 → 24 hrs.
$T = 180^\circ$ (summer solstice) → 270° (autumnal equinox), $\delta > 0$, daylength = 24 → 12 hrs.
$T = 270^\circ$ (autumnal equinox) → 360° (winter solstice), $\delta < 0$, daylength = 12 → 0 hrs.

Using the Earth's tilt angle of $\theta = 23.5^\circ$ and Seattle's latitude of $\varphi = 47.6^\circ$, we can plot daylength as a function of time of year T. (Note: Seattle (pop. 700,000) is of interest because among major U.S. cities it is the northernmost, giving it the broadest range of daylength.) The graph below shows how our calculated values correspond to "actual" values (available at timeanddate.com and presumably also calculated based on a formula) for the one-year period from December 21, 2015 to December 21, 2016.



The actual values are systematically about 2% larger than our calculated values. This discrepancy arises because our calculation does not correct for secondary effects such as the size of the Sun (timeanddate.com defines sunrise and sunset as the times when the upper edge (not the center) of the Sun's disc touches the horizon) combined with atmospheric refraction occurring when sunlight encounters air rather than the virtual vacuum of outer space.

Refraction bends rays of the Sun as they pass through the Earth's atmosphere. This bending is towards the Earth, making sunrise a little earlier and sunset a little later than would occur in a vacuum. Therefore, actual daylight hours are a little *longer* than they would be if both the Sun were a point of light (rather than a fiery ball with spatial extent) and our Earth had no atmosphere.

We could go on to improve our calculation by accounting for such secondary effects, but since our present calculation has already run on for more than a daylength, let's leave such fiddling for some other day.