Derivation of Polar Reduction Formula for a Calculator

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1 Problem Statement

The polar reduction formula takes an observer's assumed position (*lat, lon*), and a body's celestial coordinates (*declination, local hour angle*) and finds the altitude and azimuth of the body:

$$\begin{aligned} \langle r, \theta \rangle &= \mathrm{pol}(\tan dec \cos lat - \sin lat \cos lha, -\sin lha) \\ Hc &= \cos^{-1}(r \cos dec) \\ Zn &= \theta \bmod 360° \end{aligned} \qquad \text{(eq 1)}$$

In the following, we will derive this result.

2 Approach

Computing the altitude and azimuth of a body from its declination and local hour values may be viewed as translating between two different coordinate systems. We will begin by specifying the two spherical coordinate systems, the equatorial and the horizon systems. In these systems, the position of a body is designated by angles e.g. altitude and azimuth for the horizon system.

Next, we introduce rectangular coordinate systems for each of the spherical systems. We do this, because we have a way to translate coordinates in one rectangular system to the coordinates in another rectangular system.

Finally, we translate from rectangular coordinates back to spherical coordinates in the destination system.

In summary, we will do:

spherical equatorial (*dec, lha*) → rectangular equatorial (*x, y, z*) →

rectangular horizon (*x′, y′, z′*) → spherical horizon (*alt, azimuth*).

3 Background: Polar/Rectangular Systems in 2-D

3.1 Introduction

Before proceeding, we will discuss the two dimensional case because, first, it is good background for the three dimensional case, and second, because the pol function used in (eq 1) transforms 2-D rectangular coordinates to polar coordinates, and we will want to know some details about it.

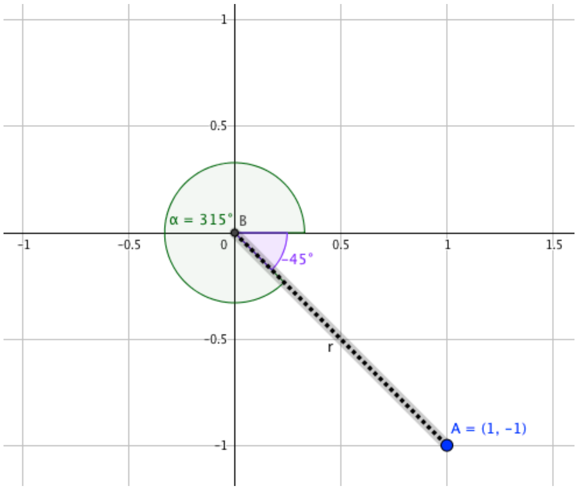


Figure 1

In Figure 1, point A is designated in polar coordinates by 1) a straight line distance from the origin (*0, 0*), and 2) an angle measured positively counter-clockwise from the horizontal *x*-axis. As shown, A has polar coordinates $\left(\sqrt{2}\,, 315°\right)$ or $\left(\sqrt{2}\,, -45°\right)$. That is, point A is a distance 1.414 from the origin, and rotated 315° counter-clockwise from the positive *x*-axis (or equivalently, -45°, which is 45° clockwise). We note that electronic calculators such as the CASIO *fx*-300MS or TI-30X restrict the angle to the range [-180°, +180°], and thus would display -45° for the angle of point A.

In the rectangular coordinate system, point A is 1 unit along the *x*-axis and -1 (i.e. "down") along the *y*-axis, and thus has coordinates (1, -1). By convention, the *x* value always goes first, and the *y* value second in the (*x, y*) coordinate pair.

3.2 Polar to Rectangular Conversion

$$(r,\theta) \rightarrow (x,y)$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad \text{(eq 2)}$$

When doing this conversion, calculators will accept any angle, not just those between negative 180° and plus 180°. We usually restrict *r* to positive values (a distance from the origin), but (-*r*, *θ*) may be interpreted as (|*r*|, *θ* +180).

3.3 Rectangular to Polar Conversion

$$(x,y) \rightarrow (r,\theta)$$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \alpha &= \tan^{-1}\left(\left|\frac{y}{x}\right|\right) \end{aligned} \qquad \text{(eq 3)}$$

$$\alpha' = \begin{cases} \alpha & \text{if } x \geq 0 \\ 180 - \alpha & x \text{ negative} \end{cases}$$

$$\theta = \begin{cases} \alpha' & \text{if } y \geq 0 \\ -\alpha' & y \text{ negative} \end{cases}$$

Given the (*x, y*) coordinates, the distance from the origin immediately follows from the Pythagorean Theorem. By definition, the angle is between the position vector and the <u>*x*-axis</u>; and the tangent of the angle is the opposite side *y* divided by the adjacent side *x*. However, to ensure the angle is in the range -180° to +180°, we apply the rules as shown. The first step is to use the absolute value of *y*/*x*, i.e. a positive number, and then use the signs of *x* and *y* to adjust the result.

The reason we are going into such detail is to examine what occurs if we swap the *x* and *y* coordinates, a trick we will use below in our derivation of (eq 1).

3.4 Polar Conversion: swapping x and y

Assume we use a CASIO calculator, and enter the (*x, y*) values in the opposite order: pol(*y, x*). Per (eq 3), the *r* value will be computed correctly, as the order of *x* and *y* does not matter. However, the angle is a different story. The angle α in (eq 3) is formed by the *y*-axis and the position vector, and is computed as the inverse tangent of <u>opposite side *x*</u> divided by <u>adjacent side *y*</u>. This is shown in Figure 2.

Figure 2

Now when we apply the rules (remembering *x* and *y* are swapped), we find θ=135°, which is the angle measured clockwise from the positive *y*-axis. We take note that <u>this is how we measure azimuth</u>, clockwise from due north, with the difference that azimuth is measured continuously clockwise from 0° to 360° and angle θ is negative (i.e. measured counter-clockwise from the positive *y*-axis) for angles greater than 180°. We will fix this "bug" by adding 360° to negative θ to bring it into our desired range. But we are getting ahead of ourselves. Let's now introduce the 3-D spherical coordinate systems.

4 Two Spherical Coordinate Systems

4.1 Equatorial Coordinate System

In this system (see Figure 3), the <u>origin</u> is at the <u>center of the earth</u>, the *x*-*y* plane lies in the plane of the equator, and the earth spins about the *z*-axis, which points to the north pole. The *x*-axis lies in the equatorial plane, and points to the observer's meridian.

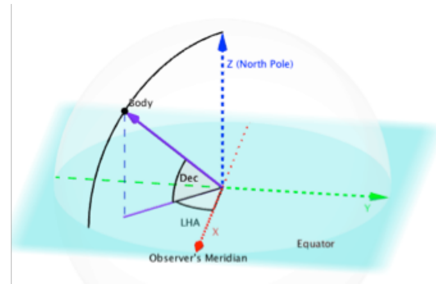


Figure 3

The spherical coordinates (usually only the angles are given) are:

1. the declination (*dec*) angle above (positive) or below (negative) the equatorial plane (equivalent to latitude);
2. the local hour angle (*lha*), measured <u>positively westward</u> on the equator from the observer's longitude, 0° to 360°;
3. distance from the center, which is understood to be fixed at *r*=1.

The rectangular coordinate system will be right-handed, origin at the center of the earth, with orthogonal axes that <u>point in the given directions</u>:

1. *x*-axis - points at the intersection of the observer's meridian and the equator;
2. *y*-axis - lies in the equatorial plane and points eastward relative to the observer;
3. *z*-axis - points to the earth's north pole.

4.2 Horizon Coordinate System

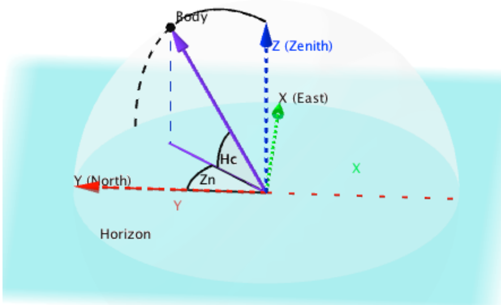


Figure 4

In the horizon system (Figure 4), the origin is at the observer, in the plane of the horizon. The *z*-axis points directly overhead.

The spherical coordinates are:

1. Altitude (*Hc*) - the angle above the horizon. It is possible to measure small negative altitudes due to refraction of the atmosphere;
2. Azimuth (*Zn*) - the angle measured clockwise (or eastward) from due north, continuously 0° to 360°;
3. distance from the observer, which is fixed at *r*=1.

The rectangular system will be right-handed, origin at the observer, with orthogonal axes:

1. *x*-axis - points east
2. *y*-axis - points north
3. *z*-axis - points directly up (the zenith)

4.3 How the Two Coordinate Systems are Related

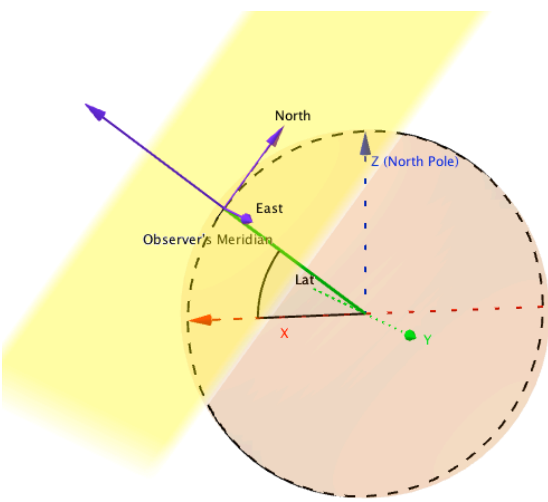


Figure 5

The local horizon system (Figure 5) is the plane tangent to the earth at the observer's position, with its *x*-axis pointing east, its *y*-axis pointing north, and its *z*-axis pointing upward, perpendicular to the plane. The angle *L* between the *z*-axis (zenith) and the plane of the equator will be the latitude of the observer.

When converting between the two systems, we will assume they share the same origin at the center of the earth. This perhaps makes more intuitive sense if we envision the earth the size of a marble, in which case the direction to some star does not measurably change if we move to the center of the marble. For close bodies, the moon and planets, we adjust ("correct") our measurements (taken at the earth's surface) to values as if taken at the earth's center.

5 Spherical to Rectangular Coordinate Transformation

5.1 Rectangular Coordinates

The three rectangular coordinate values (x, y, z) are "instructions" on how to reach a point when starting at the origin. They specify in order, the distance to move in the x direction (a negative number means move in the opposite direction of x), in the y direction, and in the z direction, as shown in Figure 6.

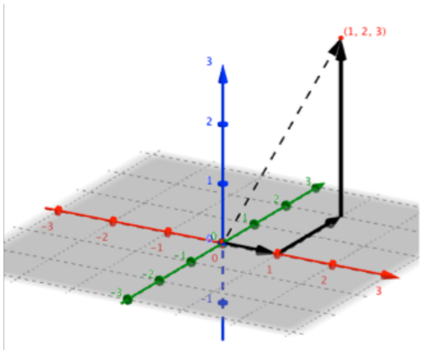


Figure 6

5.2 Spherical Equatorial to Rectangular Equatorial

Given a point identified by spherical coordinates (dec, lha) (see Figure 3), we use trigonometry to find the distance to move in the z direction:

$$\sin dec = \frac{opposite}{hypotenuse} = \frac{z}{1} = z \qquad \text{(eq 4)}$$

where we use the fact that the distance from the origin to the point is assumed to be one. Similarly, the length of the projection of the point onto the x-y plane is cos dec. Now analyze the triangle formed by legs along the x- and y-axes, and hypotenuse of length cos dec.

$$\cos lha = \frac{adjacent}{hypotenuse} = \frac{x}{\cos dec} \Rightarrow x = \cos dec \cos lha \qquad \text{(eq 5)}$$

In the same way, we find the distance to move along the y-axis, but with this observation, the direction is <u>opposite to the y-axis positive direction</u>, hence the distance is negated:

$$\sin lha = \frac{opposite}{hypotenuse} = \frac{-y}{\cos dec} \Rightarrow y = -\cos dec \sin lha \qquad \text{(eq 6)}$$

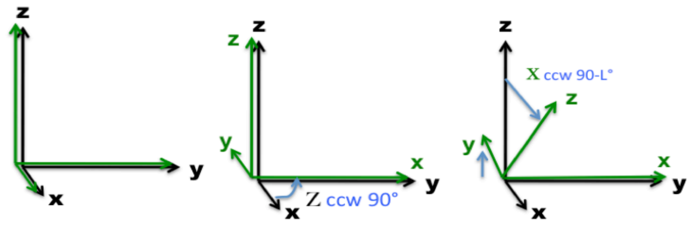
Collecting these values into a 3 x 1 vector, we have

$$\boxed{\begin{array}{l}\text{Equatorial Spherical } (dec,\ lat) \rightarrow \text{Rectangular } (x,\ y,\ z) \\ \quad \mathbf{p} = \begin{bmatrix} \cos dec \cos lha \\ -\cos dec \sin lha \\ \sin dec \end{bmatrix}\end{array}} \qquad \text{(eq 7)}$$

5.3 Spherical Horizon to Rectangular Equatorial

By a similar process to the previous section, we have (see Figure 4):

$$\boxed{\begin{array}{l}\text{Horizon Spherical } (Hc,\ Zn) \rightarrow \text{Rectangular } (x,\ y,\ z) \\ \quad \mathbf{q} = \begin{bmatrix} \cos Hc \sin Zn \\ \cos Hc \cos Zn \\ \sin Hc \end{bmatrix}\end{array}} \qquad \text{(eq 8)}$$



Intrinsic Rotations Black to Green

Figure 7

6 Equatorial Rectangular to Horizon Rectangular

6.1 Rotations

The next step is to find the rotation matrix that takes us from the equatorial rectangular system to the horizon rectangular system as shown in Figure 7. The mathematics is not generally taught in high school, so I provide some discussion in a separate document. We find the rotation matrix to be:

$$\mathbf{B} = \begin{bmatrix} 0 & -\sin L & \cos L \\ 1 & 0 & 0 \\ 0 & \cos L & \sin L \end{bmatrix} \text{, } L\text{= latitude of the observer} \qquad \text{(eq 9)}$$

and consequently, the equation* to convert equatorial coordinates to horizon coordinates is:

$$\mathbf{q} = \mathbf{B}^{\mathrm{T}}\mathbf{p} \qquad \text{(eq 10)}$$

where **p** are the rectangular equatorial coordinates, and **q** are the corresponding rectangular horizon coordinates.

Substituting into (eq 10) the definitions of (eq 7), (eq 8), and (eq 9):

$$\begin{bmatrix} \cos Hc \sin Zn \\ \cos Hc \cos Zn \\ \sin Hc \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sin L & 0 & \cos L \\ \cos L & 0 & \sin L \end{bmatrix} \begin{bmatrix} \cos dec \cos lha \\ -\cos dec \sin lha \\ \sin dec \end{bmatrix} \qquad \text{(eq 11)}$$

Multiplying out (eq 11), the result of the right-hand side are the rectangular coordinates of the point in terms of the horizon system, and thus must equal the same coordinates as computed using the spherical coordinates of the horizon system:

$$\begin{bmatrix} \cos Hc \sin Zn \\ \cos Hc \cos Zn \\ \sin Hc \end{bmatrix} = \begin{bmatrix} -\cos dec \sin lha \\ \cos L \sin dec - \sin L \cos dec \cos lha \\ \sin L \sin dec + \cos L \cos dec \cos lha \end{bmatrix} \qquad \text{(eq 12)}$$

In (eq 12), we can equate corresponding components, and in particular, we note

$$\sin Hc = \sin dec \sin L + \cos dec \cos L \cos lha \qquad \text{(eq 13)}$$

which is the formula, derived from spherical geometry, often used to solve for the altitude *Hc*. However, we will use a different approach.

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* See my *Rotation Matrices* paper for details.

7 Rectangular Horizon to Spherical Horizon

7.1 Transformation using 3-D components

Examining the rectangular components of (eq 8), we see:

$$\mathbf{q} = \begin{bmatrix} \cos Hc \sin Zn \\ \cos Hc \cos Zn \\ \sin Hc \end{bmatrix} \Rightarrow \begin{cases} \dfrac{x}{y} = \dfrac{\cos Hc \sin Zn}{\cos Hc \cos Zn} = \tan Zn \\ z = \sin Hc \end{cases} \qquad \text{(eq 14)}$$

and thus, given the (*x, y, z*) values, we can compute the corresponding spherical angles *Hc* and *Zn*.

7.2 Transformation using 2-D components

Now we use some cleverness* to note that the 2-D vector, with entries (*y, x*) i.e. the first two components of **q**, but swapped:

$$\mathbf{v} = \begin{bmatrix} \cos Hc \cos Zn \\ \cos Hc \sin Zn \end{bmatrix} \qquad \text{(eq 15)}$$

has some interesting properties. Using (eq 3) to convert from rectangular to polar coordinates, we find that the length is:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{\cos^2 Hc \cos^2 Zn + \cos^2 Hc \sin^2 Zn} \\ r &= \cos Hc \sqrt{\cos^2 Zn + \sin^2 Zn} \\ r &= \cos Hc \end{aligned} \qquad \text{(eq 16)}$$

and the angle is

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sin Zn}{\cos Zn}\right) \\ \alpha &= \tan^{-1}(\tan Zn) = Zn \end{aligned} \qquad \text{(eq 17)}$$

As noted in Section 3.4, swapping the *x* and *y* components of the **q** vector also swaps the "reference axis" from *x* to *y* (i.e. to the horizon axis pointing north) and the positive angular direction from counter-clockwise to clockwise (due to the sign rules we use).

Once we have this nice trick, we can apply it to the transformed vector of (eq 12). The first two components, swapped, give us the vector of (eq 15):

$$\mathbf{v} = \begin{bmatrix} \cos L \sin dec - \sin L \cos dec \cos lha \\ -\cos dec \sin lha \end{bmatrix} \qquad \text{(eq 18)}$$

and its length *r*= cos *Hc*, and its angle θ= *Zn*. (Remember -- the reader may verify this with calculation -- the vectors of (eq 12) represent the same point, and are numerically equal).

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* I am using Robin Stuart's result, http://fer3.com/arc/m2.aspx/Which-calculator-use-for-arctantan269-Stuart-feb-2017-g38110

7.3 A simplification

We can slightly simplify the calculation of (eq 18) by factoring out cos *dec* from each component (note tan *dec* = sin *dec*/cos *dec*, thus sin dec= cos *dec* tan *dec*),

$$\mathbf{v} = \cos dec \begin{bmatrix} \tan dec \cos L - \sin L \cos lha \\ -\sin lha \end{bmatrix} \qquad \text{(eq 19)}$$

and define a new vector **w**:

$$\mathbf{w} = \begin{bmatrix} \tan dec \cos L - \sin L \cos lha \\ -\sin lha \end{bmatrix} \qquad \text{(eq 20)}$$

The angle of **w** remains unchanged; it is *Zn* (in the range -180° to +180°), but its length must be multiplied by cos *dec* to get the original length of **v**.

Equation (eq 20) represents our key result.

7.4 Ambiguity of the Sign of the Altitude *Hc*

This approach can not distinguish altitudes below the horizon, e.g. the sun at sunrise. When the sign is not certain, the observer may compute the altitude in the traditional way, with (eq 13). Another way is to examine the sign of the *z*-component of (eq 12), or after factoring out cos *dec* (always a positive term):

$$\tan dec \sin L + \cos L \cos lha \qquad \underline{\text{if negative, } Hc \text{ is negative}} \qquad \text{(eq 21)}$$

This term is related to the first term of vector **w** (eq 20), and perhaps can be memorized:

$$\begin{array}{l} \tan dec \cos L - \sin L \cos lha \\ \tan dec \sin L + \cos L \cos lha \end{array} \qquad \text{(eq 22)}$$

8 Conclusion

From (eq 20), and an understanding of rectangular to polar conversions as implemented by a scientific calculator, we arrive at our result:

$$\langle r, \theta \rangle = \mathrm{pol}(\tan dec \cos lat - \sin lat \cos lha, -\sin lha)$$
$$Hc = \cos^{-1}(r \cos dec)$$
$$Zn = \theta \bmod 360°$$

Here, θ mod 360° is short-hand for "adjust θ to be in the range 0° to 360°"; specifically, add 360° if θ is negative.

In summary, the conveniences of this method are:

- Only one complicated formula to remember and key in.
- The altitude *Hc* and azimuth *Zn* are computed at the same time.
- There are no complicated "sign rules" to remember to find the azimuth, other than to add 360° if the result is negative.

One caveat: possible ambiguity of the sign of *Hc* if the body is near the horizon.