Martelli's Time-Sight Tables

One of the standard equations used in the solution of the navigation triangle is

$$\sin h = \sin d \sin L + \cos d \cos L \cos t \qquad (1)$$

where h is the body's observed altitude above the horizon, d is the body's declination, L is the latitude of the observer, and t is the meridian angle, assumed to be in the range (-90º to +90º).

Solving (1) for cos t, we have

$$\cos t = \frac{\sin h - \sin d \sin L}{\cos d \cos L} \qquad (2)$$

Equation (2) can be cast into a form that uses the haversine

$$\operatorname{hav} x = \frac{1}{2}(1 - \cos x) \qquad (3)$$

by multiplying both sides of equation (2) by - ½ , and adding ½

$$\begin{aligned} \frac{1}{2} - \frac{\cos t}{2} &= \frac{1}{2} - \frac{\sin h - \sin d \sin L}{2 \cos d \cos L} \\ \frac{1}{2}(1 - \cos t) &= \frac{\cos d \cos L - \sin h + \sin d \sin L}{2 \cos d \cos L} \end{aligned} \qquad (4)$$

The left-hand side of (4) is the haversine function. The right-hand side can be simplified using the *difference of angles formula* for the cosine:

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \qquad (5)$$

Using (5) in (4) we find Martelli's starting formula:

$$\operatorname{hav} t = \frac{\cos(L \sim d) - \sin h}{2 \cos d \cos L} \qquad (6)$$

The operator ~ means subtract the smaller value from the larger if the latitude and declination have the "same name", otherwise, add the two values. These rules follow from the identity

$$\cos(-x) = \cos x$$

and the convention that North (latitude or declination) is a positive angle and South is a negative angle.

For whatever reason, Martelli inverted (6), and developed his tables based on

$$\boxed{\frac{1}{2 \operatorname{hav} t} = \frac{\cos d \cos L}{\cos(L \sim d) - \sin h}} \qquad (7)$$

APPROACH

Martelli uses logarithms base ten to replace the multiplication and division operations in (7) with additions of logarithms. Taking the log of both sides of (7) we have

$$\begin{aligned} \log\left(\frac{1}{2 \operatorname{hav} t}\right) &= \log\left(\frac{\cos d \cos L}{\cos(L \sim d) - \sin h}\right) \\ \log 1 - \log 2 - \log \operatorname{hav} t &= \log(\cos d) + \log(\cos L) - \log(\cos(L \sim d) - \sin h) \end{aligned} \qquad (8)$$

TABLE I *LOG OF LAT. AND DECLINATION*

Table I is used to find the logarithm base ten of the cosines of the declination and the observer's latitude. The cosine of angles between 0º and ±90º are between one and zero. The log of a number less than one is negative (and undefined for zero). Martelli did not want to subtract logs (adding positive numbers is much less error prone), so he limited the angles (acceptable declination or latitude) to the interval 0º to 71º 34' and added an offset to the logs to make all the values positive. The cosine of the upper limit is 0.3162, and the log of this value is -0.5 to four decimal places. He adds 0.5 to the $\log_{10}(\cos(x))$ (to ensure it is positive), scales the result by 1e4 (10,000) and rounds to the nearest integer to make the entries 4 digit numbers.

TABLE I $$1e4\cdot\left(0.5+\log_{10}\cos x\right) \tag{9}$$

x is L (latitude) or d (declination). Range is 0º to 71º 34' The entries on the first page run across from 0º to 11º, and down the page 0' to 59'. The minutes are in groups of five for ease of reading. The last page gives 60º to 71º. TABLE I has a total of six pages.

TABLE II *SUM OR DIFFERENCE*

Martelli's second table performs a cosine lookup on L~d and responds with an entry in the form minutes seconds and tenths of seconds. I do not see the advantage of this format, and it does introduce a source of error when adding seconds and incorrectly "carrying" multiples of sixty seconds into the minutes column.

TABLE II $$a=\frac{50}{3}\cos\left(L\sim d\right)+\frac{10}{3} \tag{10}$$

Output is in the form mm ss.s (minutes, seconds and tenths of seconds). For example, with L~d= 60º, the table entry is 11' 40.0" (representing the value 11.66667). L~d means add if opposite names and subtract if same names. Treat all values as positive. The first page goes across 0º to 9º. Minutes 0 to 59 run down the page, in groups of six. The last page is 80º to 89º. TABLE II has a total of nine pages.

TABLE III *ANGLE OF ALTITUDE*

The third table is the sine lookup, with scaling.

TABLE III $$b=\frac{50}{3}\left(1-\sin h\right) \tag{11}$$

Output is identical to TABLE II, in the form mm ss.s, as is its layout. There are nine pages.

Then respondents of TABLE I and TABLE II are added together

$$c=a+b \tag{12}$$

to form a value expressed as minutes, seconds and tenths of seconds.

TABLE IV *AUXILIARY LOGARITHM*

The fourth table applies the $\log_{10}$ to the summation c given by (9).

$$\text{TABLE IV} \qquad d = 1e4\log\left(\frac{36}{c-20}\right) \tag{13}$$

The argument c to Table IV is in minutes, seconds and tenths of seconds. The first page is 20', with 0" to 59" running down the page (in groups of six), and tenths of seconds 0.0" to 0.9" going across. Entries are five digits, with the leading digit separated by a period from the trailing four digits. The last page is 36'. Table IV has seventeen pages.

TABLE V *LOG OF HOUR ANGLE*

The last table extracts the inverse haversine, and then divides the angle t by fifteen to convert it from degrees to hours, minutes and seconds. This could be modified to return degrees instead.

$$\text{TABLE V} \qquad f = 1e4\log_{10}\left(\frac{21.6}{1-\cos(15t)}\right) \tag{14}$$

The entries f in Table V are computed by using this expression, evaluated for t= 16 hours up to 24 hours in increments of five seconds. (Or for t=8 hours down to 0 hours in decrements of five seconds.) This can be modified to use degrees by replacing $15t$ with a (for angle), and evaluating the expression for a= 120º to 360º (or -120º to 0º) in increments of one arc minute.

There is one hour per page. The label on the pages are:

4 :11 and (4:11)+12 at the top of the page, labeled A.M.

11-(4:11) at the bottom of the page, labeled P.M.

For example, the second page is labeled at the top, "5 or 17 Hours, A.M" and at the bottom, "6 Hours, P.M." (Personally, I would drop the 17 from the top label.) Minutes 0 to 59^m run down the page, in groups of six. Seconds run across the top of the page 00" by 5" to 55". At the bottom of the page, the seconds label run in the reverse order: 60" by -5" to 05". The first column shows five digits, and the remaining columns only the lowest three digits.

CONSISTENCY WITH EQUATION (6)

We can use the definitions for the entry in each table to show that they are consistent with the original "time-sight" equation given by (6).

The first step in the procedure is to use Table I with argument L and d. These respondents are added:

$$1e4\{\log\cos d + \log\cos L + 1\} \tag{15}$$

The table entries include a 0.5 for each term, and the results are scaled.

Second, the respondents from Table II and III are added:

$$c = \frac{50}{3}\cos(L \sim d) + \frac{10}{3} + \frac{50}{3}(1 - \sin h)$$
$$c - 20 = \frac{50}{3}(\cos(L \sim d) - \sin h) \tag{16}$$

Third, Table IV adds to (15) the term

$$1e4\log\left(\frac{36}{c - 20}\right) = 1e4\{\log 36 - \log(c - 20)\} \tag{17}$$

Substituting the definition of c-20 given by (16) into (17) we have

$$1e4\left\{\log 36 - \log\left(\frac{50}{3}(\cos(L \sim d) - \sin h)\right)\right\}$$
$$1e4\left\{\log 36 - \log\frac{50}{3} - \log(\cos(L \sim d) - \sin h)\right\} \tag{18}$$

The expression (18) is added to the terms in (15), and the logarithms of the numerical values combined, to get

$$f = 1e4\{\log\cos d + \log\cos L - \log(\cos(L \sim d) - \sin h) + \log 21.6\} \tag{19}$$

Fourth, the value of (19) is the argument to Table V. We have the equality

$$1e4\log_{10}\left(\frac{21.6}{1 - \cos(15t)}\right) = 1e4\{\log\cos d + \log\cos L - \log(\cos(L \sim d) - \sin h) + \log 21.6\} \tag{20}$$

Cancel the scaling factor from both sides, and combine the log expressions on the right-hand side we have

$$\log_{10}\left(\frac{21.6}{1 - \cos(15t)}\right) = \log_{10}\left(\frac{21.6\cos d\cos L}{\cos(L \sim d) - \sin h}\right) \tag{21}$$

Take the antilog of both sides, cancel the 21.6 factor, invert both sides, and divide by 2:

$$\left(\frac{1 - \cos(15t)}{2}\right) = \frac{\cos(L \sim d) - \sin h}{2\cos d\cos L}$$
$$\text{hav}15t = \frac{\cos(L \sim d) - \sin h}{2\cos d\cos L} \tag{22}$$

This is the "time-sight" formula (6), with argument t in hours.