The Nautical Almanac's Concise Sight Reduction Tables

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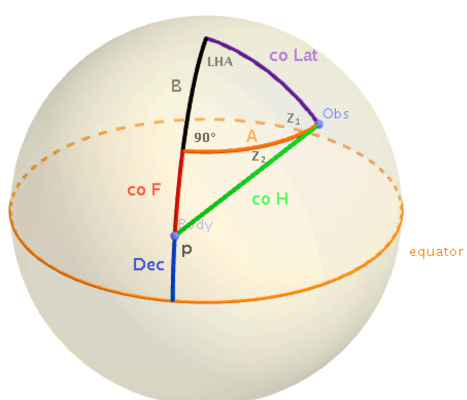
This document describes the mathematics used to derive the Nautical Almanac's Concise Sight Reduction Tables (NACSR). These tables, at the back of the Nautical Almanac, provide a compact (31 pages) set of tables that enable the reduction of celestial data to intercept and azimuth angle from an assumed position. This data represents a Line of Position that can be plotted. The intersection of multiple LOPs provides the celestial fix.

The NA gives a short description:

> Entries in the reduction table are at a fixed interval of one degree for all latitudes and hour angles. A compact arrangement results from division of the navigational triangle into two right spherical triangles, so that the table has to be entered twice.

Background



This paper will consider only the case of the Local Hour Angle from zero to ninety degrees, both the observer and the celestial body having the "same name" (in the Northern Hemisphere), and the declination is less than the observer's latitude. See Figure 1. The various other cases can be examined in a similar manner to this case.

Figure 1: The Navigational Triangle, with vertices at the North Pole, at the location of the observer, and at the "ground position" of the body.

Definitions:
Side - an arc along a great circle, measured in degrees.
co x - given a measurement x in degrees, co x means 90º - x.

Sides of the Navigational Triangle

1. co Lat - the observer is at latitude Lat, measured in degrees north of the equator. The arc from the observer to the North Pole is 90º- Lat, i.e. co Lat.
2. co Dec - the body is at latitude Dec, and the angular distance from the body's geographic position to the North Pole is co Dec.

3. co H - H is the observed altitude of the body above the horizon. The arc, measured in degrees, from the observer to the ground position of the body is 90º - altitude (i.e. the zenith distance), or co H.

Angles of the Navigational Triangle

1. LHA - Local Hour Angle - measured at the North Pole. It is the difference in degrees between the longitudes of the body and the observer. By convention, it is measured clockwise (looking down at the North Pole) from the observer to the body.
2. Z - azimuth angle - measured at the observer.
3. co p - measured at the body - an angle unused in navigation.

Divide the Navigational Triangle and Name the New Sides

The triangle is divided into two right triangles by dropping a "perpendicular" from the observer to the (possibly extended) side labeled co Dec. This side is labeled A in Figure 1.

The "top" triangle will have a side denoted B, from the North pole to where side A intersects the longitude of the body. The components of the top triangle are: (1) side B, (2) angle 90º, (3) side A, (4) angle Z_1, (5) side co Lat, and (6) angle LHA.

To label the bottom triangle, we define

F= B+Dec. (This will probably follow "same name, contary name" conventions.)

One side of the bottom triangle is 90 - B - Dec, or in other words, co F. The components of the bottom triangle are: (1) side A, (2) angle 90º, (3) side co F, (4) angle co p, (5) side co H, (6) angle Z_2.

Apply Napier's Rules

Reference https://en.wikipedia.org/wiki/Spherical_trigonometry#Napier.27s_rules_for_right_spherical_triangles

Napier's Rules use a pentagon as a memory aid to various trigonometric relations of a right spherical triangle:

sine (middle) = product of the cosine(opposites)

sine (middle) = product of the tangent(adjacents)

The pentagon is filled with the components as listed above, entered clockwise in the given order, but ignoring the 90º angle. Also, each of the components (4), (5) and (6) have a "co" applied to it when entered into the pentagon. Note that co co x becomes x.

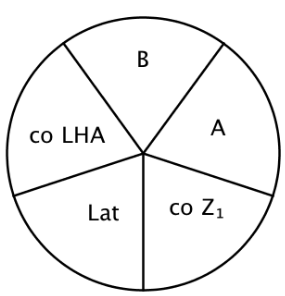


Figure 2 - Top Triangle's Napier Pentagon

Using the first Napier Rule,

sine (middle) = product of the cosine(opposites),

and the identities:

sin(co x) = cos(x); cos(co x) = sin(x);

we can write the following relations:

1. sin A= cos Lat sin LHA
2. cos LHA = sin Z_1 cos A; $\sin Z_1 = \frac{\cos LHA}{\cos A}$
3. sin B = sin Z_1 cos A

These relations are used to develop a table with arguments Lat and LHA (both in whole degrees). The table responds with:

1. $A = \sin^{-1}(\cos Lat \sin LHA)$
2. $Z_1 = \sin^{-1}\left(\frac{\cos LHA}{\cos A}\right)$
3. $B = \sin^{-1}(\sin Z_1 \cos A)$

rounded to the nearest arc minute (1/60th of a degree).

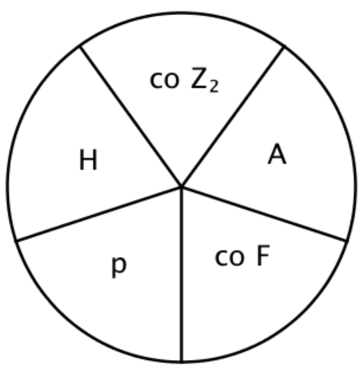


Figure 3 - Bottom Triangle's Napier Pentagon

Similarly, for the bottom triangle we can write:

4. sin H= cos A sin F , where F= B+Dec
   also 4a): $\sin F = \frac{\sin H}{\cos A}$
5. cos F = cos H sin Z_2 from which we get
   5a): $\sin Z_2 = \frac{\cos F}{\cos H}$
6. sin p = sin Z_2 cos A and using 5a):
   $\sin p = \frac{\cos F}{\cos H}\cos A$

We have the exact same relations as for the top triangle, namely, entering with (whole number) arguments A and F, the table responds with

1. $H = \sin^{-1}(\cos A \sin F)$
2. $Z_2 = \sin^{-1}\left(\frac{\cos F}{\cos H}\right)$
3. $p = \sin^{-1}(\sin Z_2 \cos A)$

Notice that the arguments to the table and the respondents from the table are of identical form for both the top and bottom triangles. That is, we only need this one table to compute H (i.e. H_c) and the azimuth angle $Z= Z_1+Z_2$. The first time we use the table we can adjust the arguments to be whole values (by picking a suitable "assumed position"). The table responds with values A and B. The user adds B to the known (i.e. from the Almanac) value Dec to compute F= B+Dec. Both A and F are to the nearest *minute*, and must be rounded to the nearest *degree* in order to use the table. This means the respondent H from the second use of the table will have some error due to rounding both A and F. Of course, Z_2 will also have some error, but we can (for H less than 80º) ignore this error. But we do need H to be accurate to at least a minute or so. We can correct for the error in H in the following way.

Correction Formula

Beginning with 4. sin H= cos A sin F , use implicit differentiation to compute the total differential:

$$d\sin H = d\cos A \sin F$$

$$\cos H \; dH = \cos A \cos F \; dF - \sin A \sin F \; dA \qquad \text{Eq 1}$$

$$dH = \frac{\cos A \cos F}{\cos H} dF - \frac{\sin A \sin F}{\cos H} dA$$

Replace the infinitesimals dH, dF and dA to get the approximation formula

$$\Delta H = \frac{\cos A \cos F}{\cos H} \Delta F - \frac{\sin A \sin F}{\cos H} \Delta A \qquad \text{Eq 2}$$

From item 6 in the previous section we can replace the first term of Eq 2 with sin p. Also, using 4a), we can replace sin F in the second term. We get

$$\Delta H = \sin p \; \Delta F - \frac{\sin A \sin H}{\cos A \cos H} \Delta A$$

$$\Delta H = \sin p \; \Delta F - \tan A \tan H \; \Delta A \qquad \text{Eq 3}$$

By Napier's 2nd rule, and using the bottom triangle's pentagon, we find

$$\sin(90 - Z_2)= \tan A \; \tan H \qquad \text{Eq 4}$$

and the *Correction Formula*:

$$\Delta H = \sin p \; \Delta F - \sin(90 - Z_2) \; \Delta A \qquad \text{Eq 5}$$

Here is how we use this formula. The first time we use the main table, it responds with B, from which we find F_{true} (for lack of a better term)= B+Dec. But to use the table a second time, we round F_{true} to the nearest whole degree, and thus F_{true} = F + ΔF. If we round up, ΔF will be negative, and if we round down, ΔF will be positive. For example, if F_{true} = 40º 45' then F = 41º and we should expect the H from the table to be too large. The error ΔF = -15', and the correction will be -15 * sin p, a negative quantity (we expect $0 \le p \le 90$, and thus $\sin p \ge 0$). Thus the correction table will have argument p (integral degrees) and argument ΔF in minutes. Rather than make the user find 45-60= -15, enter with F' = 45' but compute (60-F') * sin p. Make the sign of the correction negative. On the other hand, if we rounded down, H will be too small and we will want to add the correction. Example: F_{true} = 40º 15' , F= 40º , ΔF = 15' and F' also equals 15'. The arguments will be p and F', and the correction will be 15' * sin p. This will be the same value as for F'= 45', but now with a positive sign.

The second term in the correction formula is of the same form as the first, but notice it has a negative sign in front of it. This means A' from 0' to 29' will be a negative correction, and 30' to 59' a positive correction (opposite the conventions for F'). Also, the correction term is sin (90 - Z_2), which can be accomplished by entering the table from the bottom up. For example, when Z_2=90, this corresponds to argument p = 0, and when Z_2=0, to p= 90.

Thus both terms in the correction formula can make use of the same table, if we are careful about the sign conventions.

What remains is to complete the analysis for the other possible configurations that can arise, i.e. the latitude and declination have contrary names, for LHA>90º, for the South Pole being the raised pole, etc. Of course, based on the tables in the NA, we see that there must be symmetries that reduce all of these cases to a similar form that uses this same table.