# Rotation Matrices and Rotated Coordinate Systems

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Rotated Coordinate Systems is a confusing topic, and there is no one standard or approach[1]. The following provides a simplified discussion.

## Rotating a point in two-dimensions

We can rotate a point in the *real-imaginary* plane[2], as shown in Figure 1. Let vector **a** represent the complex number[3] **a** = $a_x + i\,a_y = r\exp(i\beta)$. Multiply by $\exp(i\phi)$, to get the rotated vector **b**= $r\exp(i(\beta+\phi))$. Writing the complex numbers in Cartesian coordinates, the operation is:

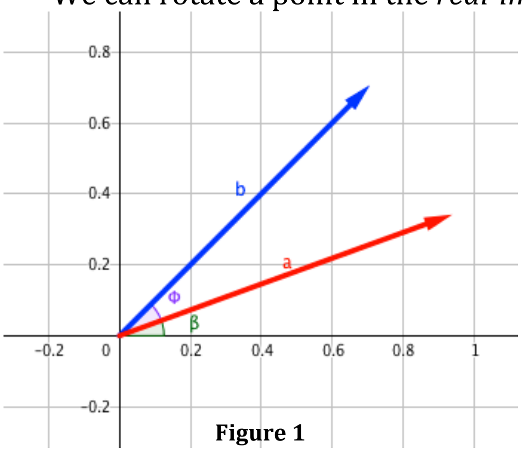


**Figure 1**

$$\left(a_x + i\,a_y\right)\left(\cos\phi + i\sin\phi\right)$$
$$= \left(a_x\cos\phi - a_y\sin\phi\right) + i\left(a_x\sin\phi + a_y\cos\phi\right)$$

This result tells us how to rotate the point (*x, y*) counter-clockwise from the *x*-axis by an angle $\phi$. Using matrix notation[4], and writing the point as a two element (vertical) vector, the rotation is written as:

$$\begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi\end{bmatrix}.$$

And thus we define the rotation matrix **R**:

$$\mathbf{R} = \begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}$$

This matrix rotates a point in the angular direction from the "first axis" (the *x*-axis) toward the "second axis" (the *y*-axis), the short way.

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1 https://en.wikipedia.org/wiki/Rotation_(mathematics)
2 https://en.wikipedia.org/wiki/Complex_number#Absolute_value_and_argument
3 https://en.wikipedia.org/wiki/Complex_number#Euler's_formula
4 https://www.mathsisfun.com/algebra/matrix-multiplying.html

**Rotate a Coordinate System**

We shall represent the basis vectors of a orthonormal coordinate system as 2-tuples, e.g. (1, 0) and (0, 1) and their equivalent 2 x 1 vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and collecting these vectors (as <u>columns</u>) in a basis matrix **B**:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ reference basis, columns are the basis vectors

It is understood that vector **x** represents a distance of "one unit" in the "*x*-direction", and similarly for the **y** vector. The directions and distance metric remain to be specified for any particular circumstance.... though the directions must be orthogonal (form a ninety degree angle).

The individual basis vectors may be rotated counter-clockwise by angle $\phi$ , by multiplying the rotation matrix **R** times each; or in one step by multiplying **RB**. However, note that some in the literature reserve **R** for rotating points. If a point is rotated by angle $\phi$, this may be viewed as the basis vectors rotating in the opposite direction, i.e. by $-\phi$, and this is equivalent to using $\mathbf{R}^T$ (**R** transpose). This is one possible point of confusion!

**Rotation Matrix in 3-Dimensions**

We will use (Figure 2) an orthogonal, Right-Handed Coordinate system[5] (RHS), and introduce the 3 x 1 **z** vector = **x** x **y**, where x represents the vector cross product[6]. If we hold the *z* coordinate constant, and rotate about the *z*-axis, counter-clockwise (from the *x*-axis toward the *y*-axis), we can use the 2-dimensional **R** matrix above embedded in a 3 x 3 rotation matrix. The convention will be:

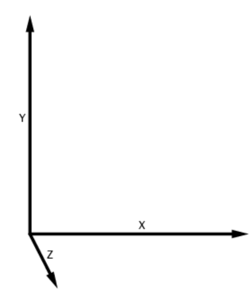



**Figure 2**

- RHS, with orthogonal axes
- Rotate about the <u>spin axis</u>, leaving two axes which will rotate. These will be ordered <u>alphabetically</u> i.e. *x, y, z*, and this order <u>defines</u> which axis of any pair will be <u>"first"</u>.
- When the observer is positioned at the tip of the spin axis, looking back toward the origin, a <u>positive angle means rotate counter-clockwise</u>. With these rules, we get the following rotation matrices:

[5] https://en.wikipedia.org/wiki/Cartesian_coordinate_system#In_three_dimensions
[6] https://en.wikipedia.org/wiki/Cross_product

$$\mathbf{Z}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{Y}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$\mathbf{X}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where the name of the rotation matrix indicates the spin axis about which the rotation occurs.

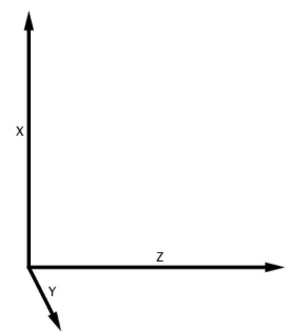


**Figure 3**

The rotation about the *y*-axis deserves a comment. Orienting the axes so that the observer is looking down on the *x-z* plane (Figure 3), we see that our 2-dimensional rotation matrix **R** will rotate from the *x*-axis toward to *z*-axis: **R** rotates the "first axis" i.e. *x* toward the "second axis" *z*, and this is <u>clockwise</u>. We would have to switch the order of the axes (make *z* the first axis) or more simply, replace θ with -θ in the matrix to achieve a counter-clockwise rotation. We chose the latter approach, and used the trigonometric identities:

$$\sin(-\theta) = -\sin(\theta), \quad \cos(-\theta) = \cos(\theta)$$

Thus in the **Y** rotation matrix, we see the signs on the sine entries negated relative to the other rotation matrices.

## Matrix Vector Multiply

Consider the columns of a 3 x 3 matrix as three 3 x 1 vectors **x**, **y**, and **z**. Examine the result of multiplying this matrix times some vector **v** with components *a*, *b*, and *c*. It can be shown that this multiplication is identical to the sum of three scaled vectors:

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$$

In particular, we note the following:

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{x}, \quad \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{y}, \quad \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{z}$$

We will use this observation below to interpret the effects of a rotation matrix.

**Chaining Rotation Matrices**

By applying to the initial coordinate system up to three sequential rotations in turn[7], we may achieve any orientation we desire. However, the topic is complicated, and we will give it (relatively) short shrift. The main idea will be to distinguish between:

1. Intrinsic rotations - use the "new" axis direction of the rotated system when we apply a second rotation.
2. Extrinsic rotations - always use the original axes directions when applying rotations.

*Intrinsic Rotations*

Consider some basis matrix **B**, where the columns are unit length vectors pointing in the directions **x'**, **y'** and **z'**. These directions may not be the original directions of the reference basis (i.e. the identity matrix **I**). Now post-multiply **B** by, for example, **Z**(θ):

$$\begin{bmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Per our observation in the previous section, the last column of **Z** will leave **z'** unchanged. Of course, both **x'** and **y'** will be rotated counter-clockwise about axis **z'**. The important observation is that the rotation occurs around the modified axis **z'** and not the original **z** axis. Thus:

*Post-multiply = intrinsic rotation = about the changing axes.*

*Extrinsic Rotations*

Now consider pre-multiplying basis matrix **B** by some rotation matrix, for example:

$$\mathbf{ZB}$$

Either matrix conforms to the requirements of being a basis or a rotation, and it is a matter of interpretation or usage that distinguishes which is which. In other words, we may interpret **Z** as a basis matrix, rotated about the *z*-axis of the original reference system. Interpreting the **B** as a rotation matrix, the multiplication applies "previous rotations" as encoded in the **B** matrix, to the original system rotated about an axis of the original reference system. We deduce:

*Pre-multiply = extrinsic rotation = about the original axes.*

[7] https://en.wikipedia.org/wiki/Euler_angles

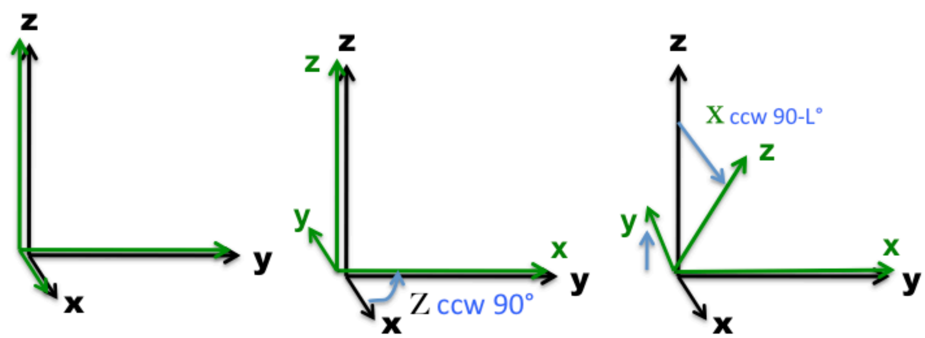
These ideas may be confusing, so let's do some examples.

**Rotating the Equatorial Coordinate System to the Horizon Coordinate System**

*Intrinsic Example*

Figure 4 shows the rotations to translate from the reference system (black) to a new system (green), that coincides with transforming from equatorial to horizon coordinate systems. Before continuing, make note that we will use these identities[8]:

$$\sin(90-x)=\cos x, \quad \cos(90-x)=\sin x$$




Intrinsic Rotations Black to Green
**Figure 4**

We begin with both systems aligned, and then rotate the green system 90° counter-clockwise about the $z$-axis. We use matrix **Z** (defined above), with theta = +90°. We get:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, rotate the green $y$ and $z$-axes (90°-L) counter-clockwise (and thus the $z$-axis is L° "above" the equatorial $x$-$y$ plane). We are doing an <u>intrinsic rotation</u>, so this means rotate about the green $x$-axis, and <u>post-multiply</u>[9]:

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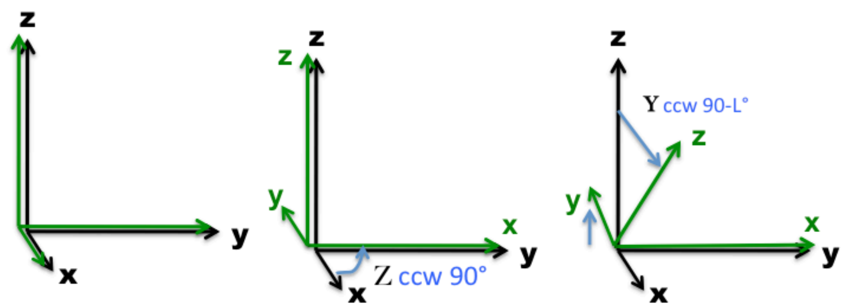
[8] https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Angle_sum_and_difference_identities
[9] https://www.mathsisfun.com/algebra/matrix-multiplying.html

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(90-L) & -\sin(90-L) \\ 0 & \sin(90-L) & \cos(90-L) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin L & -\cos L \\ 0 & \cos L & \sin L \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\sin L & \cos L \\ 1 & 0 & 0 \\ 0 & \cos L & \sin L \end{bmatrix}$$

This matrix rotates the reference system into the green system.

*Extrinsic Example*




Extrinsic Rotations Black to Green

**Figure 5**

Now let's use extrinsic rotations, where rotations are always about the axes in the reference system (Figure 5). As before, we begin with aligned systems, and rotate counter-clockwise 90° about the *z*-axis. But for the second step, we pre-multiply, and rotate 90-L about the reference (black) *y*-axis:

$$\begin{bmatrix} \cos(90-L) & 0 & \sin(90-L) \\ 0 & 1 & 0 \\ -\sin(90-L) & 0 & \cos(90-L) \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sin L & 0 & \cos L \\ 0 & 1 & 0 \\ -\cos L & 0 & \sin L \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -\sin L & \cos L \\ 1 & 0 & 0 \\ 0 & \cos L & \sin L \end{bmatrix}$$

and we get the same result as before.

### Converting between Coordinate Systems

We can use these results to translate between coordinate systems. Let a point in the reference system be identified with its 3-tuple coordinates (*x, y, z*), basis matrix **I** (the identity matrix), and its position vector:

$$\mathbf{p}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

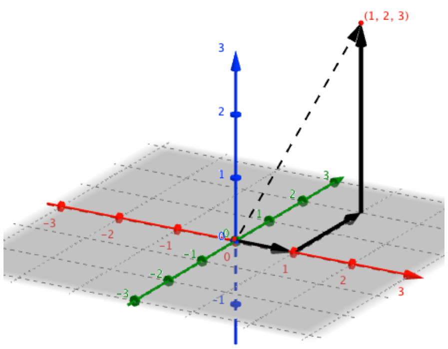



**Figure 6**

The position of this point is the scaled direction vectors (i.e. basis vectors, encoded as the columns of basis matrix **I**), added head to tail:

$$\mathbf{I}\mathbf{p}_1 = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\,\mathbf{x} + y\,\mathbf{y} + z\,\mathbf{z}$$

That same point in the rotated system with basis **B** will have different coordinates (*x′, y′, z′*), and a different position vector $\mathbf{p}_2$, and the <u>exact same location</u> is found using:

$$\mathbf{B}\mathbf{p}_2 = \begin{bmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = x'\mathbf{x}' + y'\mathbf{y}' + x'\mathbf{z}'$$

Thus, we can write the equation:

$$\mathbf{B}\mathbf{p}_2 = \mathbf{I}\mathbf{p}_1$$

where **I**, the identity matrix, is the basis matrix for the reference system and **B** is the new system.

If we know matrix **B**, this equation tells us how to translate coordinates in the rotated system to coordinates in the reference system:

$$\mathbf{p}_1 = \mathbf{B}\mathbf{p}_2$$

And vice versa[10]:

$$\mathbf{B}\mathbf{p}_2 = \mathbf{p}_1$$
$$\mathbf{B}^{-1}\mathbf{B}\mathbf{p}_2 = \mathbf{B}^{-1}\mathbf{p}_1$$
$$\mathbf{p}_2 = \mathbf{B}^{\mathrm{T}}\mathbf{p}_1$$

where we use the fact that the inverse of **B** is the <u>transpose</u>[11] of **B** (see last section).

Note that if we know the rotation matrix that transforms the reference system to the new system:

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[10] https://www.mathsisfun.com/algebra/matrix-inverse.html

[11] https://www.mathsisfun.com/algebra/matrix-introduction.html, section "Transposing"

$$\mathbf{B} = \mathbf{R}\mathbf{I}$$
$$\mathbf{B} = \mathbf{R}$$

we see that the rotation matrix is the basis matrix. And of course, $\mathbf{R}^{\mathrm{T}}$ is the matrix we use to translate reference coordinates into the new coordinates of the rotated system:

$$\boxed{\mathbf{p}_{new} = \mathbf{R}^{\mathrm{T}}\mathbf{p}_{ref}}$$

**Inverse of a Rotation Matrix is its Transpose**

By definition, rotating a position vector only modifies its direction, and never its length. Their lengths are equal:

$$|\mathbf{p}_2| = |\mathbf{p}_1|$$
$$|\mathbf{R}\mathbf{p}_1| = |\mathbf{p}_1|$$
$$|\mathbf{R}\mathbf{p}_1|^2 = |\mathbf{p}_1|^2$$
$$(\mathbf{R}\mathbf{p}_1)^{\mathrm{T}}\mathbf{R}\mathbf{p}_2 = \mathbf{p}_1^{\mathrm{T}}\mathbf{p}_1$$
$$\mathbf{p}_1^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{p}_1 = \mathbf{p}_1^{\mathrm{T}}\mathbf{p}_1$$

This must be true for all position vectors, which implies

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$$

and also

$$\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{-1}$$