ALTITUDE CORRECTIONS FOR CORIOLIS AND OTHER ACCELERATIONS

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1. Introduction. Some apology, or at least an adequate excuse, is needed for resurrecting a theoretical treatment of the effect of coriolis acceleration on observations of altitude made with a bubble sextant. Such an excuse is provided by the recent publication by Dr. J. J. Green of an article† suggesting that the correction table (Z-correction) given in both the British and American *Air Almanacs* (and in many other Air Almanacs) is incomplete. In a reasoned letter‡ to the Editor of *Navigation*, Dr. G. M. Clemence, Director of the American Nautical Almanac Office, has given a simple and straightforward explanation of the two separate and distinct causes for the deviation of the zenith as indicated by the bubble of a bubble sextant; and he has further justified the present practice adopted in the almanacs. Considerable interest has, however, been aroused and it seems opportune to give a previously unpublished general derivation of the theoretical correction, together with a brief discussion of the difficulties of practical application.

2. General description. The position on the Earth of an observer can only be determined by astronomical observation if he knows the position of his zenith on the celestial sphere. The zenith can be defined by reference to the visual horizon; but when this cannot be used, an appeal must be made to the directional force of gravity—in the form either of an artificial horizon or, in the air, of the bubble of a bubble sextant. The acceleration due to gravity will be combined, however, with all the various other accelerations of the sextant, and the zenith thus determined will be in error on this account. It is mainly for this reason that it is now the accepted practice to average altitudes observed with a bubble sextant over a period of about two minutes; in this time it is assumed that the effect of periodic and random accelerations of the aircraft, which are known to shift the apparent zenith by considerable amounts (as much as one degree), will be averaged out. Systematic deviations of the zenith, due to accelerations of constant amount and direction, will not be averaged out and should theoretically be determined and allowed for.

The so-called 'coriolis' acceleration, due to the rotation of the

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‡ *Navigation*, Vol. 1, No. 6, p. 153, 1947.

Earth, is independent of the course flown and, although much smaller than the periodic or random accelerations, is systematic; the necessity for applying a correction for coriolis acceleration to altitudes observed with a bubble sextant appears to have been first announced in 1940 by the late Thomas L. Thurlow in an unpublished paper.

Other systematic accelerations, independent of the Earth's rotation, will arise if the aircraft is following a path other than a great circle. In particular, it is Dr. Green's contention that an additional systematic correction should be made for the departure of the rhumb line from the great circle. This crucial question of whether corrections should be applied for departures from great circle courses will be considered in Section 6.

In the following Sections 3, 4 and 5 is reproduced, with only minor or verbal amendments, a general derivation of the total correction originally dated 8 June 1940 and slightly expanded for limited circulation (about six copies only) on 31 December 1942, and again on 18 January 1947.

3. Theoretical derivation. The problem is to determine the deviation of the zenith, as defined by the bubble of a bubble sextant, in an aircraft flying with given course and speed over the Earth's surface. Two simplifications of the general problem can be made immediately:

(i) It is assumed that the aircraft is flying at a constant height, and that this represents, to a sufficient approximation, a constant distance from the centre of the Earth.

(ii) It is assumed that the ground speed of the aircraft is constant.

With these assumptions the problem will be solved without any further assumption regarding the nature of the course followed by the aircraft.

The following notation is used:

a = radius of Earth, assumed constant;
ϕ = latitude (+ve in northern hemisphere);
θ = longitude (+ve for eastern longitudes);
w = angular velocity of rotation of the Earth;
V = speed, assumed constant, of aircraft;
v = angular velocity of aircraft = V/a;
A = bearing of aircraft's track, measured from the north, through east.

Derivatives with respect to the time are denoted by dashes.

The components of velocity northwards and eastwards of the aircraft, relative to the Earth, can be expressed as:

$$\text{northwards} \quad a\phi' = av\cos A$$
$$\text{eastwards} \quad a\theta'\cos\phi = av\sin A$$
$$\text{whence} \quad \phi' = v\cos A$$
$$\theta' = v\sin A\sec\phi$$

and, by differentiation,

$$\phi'' = -vA' \sin A$$
$$\theta'' = vA' \cos A \sec \phi + v^2 \sin A \cos A \sec \phi \tan \phi$$

Referred to a fixed frame of axes, instead of to the system rotating with the Earth, the transverse eastwards angular velocity, θ', is $w + v \sin A \sec \phi$, and this must be used in obtaining the accelerations. Using the standard formulæ in spherical polar coordinates (r, θ, ϕ) and making the appropriate substitutions, these accelerations are:

upwards: $r'' - r(\phi')^2 - r(\theta')^2 \cos^2 \phi$
$$= -av^2 \cos^2 A - a(w + v \sin A \sec \phi)^2 \cos^2 \phi$$

northwards: $r\phi'' + 2r'\phi' + r(\theta')^2 \sin \phi \cos \phi$
$$= -avA' \sin A + a(w + v \sin A \sec \phi)^2 \sin \phi \cos \phi$$
$$= aw^2 \sin \phi \cos \phi + 2awv \sin A \sin \phi + av^2 \sin^2 A \tan \phi - avA' \sin A$$

eastwards: $r\theta'' \cos \phi + 2r'\theta' \cos \phi - 2r\theta'\phi' \sin \phi$
$$= a(vA' \cos A \sec \phi + v^2 \sin A \cos A \sec \phi \tan \phi) \cos \phi - 2av \cos A(\omega + v \sin A \sec \phi) \sin \phi$$
$$= -2avw \cos A \sin \phi - av^2 \sin A \cos A \tan \phi + avA' \cos A.$$

The acceleration upwards can be ignored as it will merely have the effect of a small change in the value of gravity. The first term in the northwards acceleration is independent of the aircraft's speed and is, in fact, the acceleration due to the Earth's rotation which combines with the attraction of the Earth itself to give the acceleration due to gravity.

Of the remaining terms it will be seen that they can be expressed as:

northwards: $(2awv \sin \phi + av^2 \sin A \tan \phi - avA') \sin A$

eastwards: $-(2awv \sin \phi + av^2 \sin A \tan \phi - avA') \cos A$

so that the component of acceleration in the tangent plane is

$$f = 2awv \sin \phi + av^2 \sin A \tan \phi - avA' \qquad \text{I}$$

in the direction whose bearing is $270° + A$.

The values of this acceleration for certain particular paths are:

(i) For a rhumb line, the bearing of the track is constant and A' is zero; the acceleration is thus

$$2awv \sin \phi + av^2 \sin A \tan \phi \qquad \text{II}$$

(ii) For a great circle,

$$\sin A \cos \phi = \text{constant}$$

and
$$A' \cos A \cos \phi - \phi' \sin A \sin \phi = 0$$

or
$$A' = v \sin A \tan \phi$$

The acceleration is thus, simply,

$$2awv \sin \phi \qquad \text{III}$$

(iii) The acceleration is zero when

$$\sin A \cos \phi + \frac{w}{v} \cos^2 \phi = \text{constant} \qquad \text{IV}$$

for, by differentiation, we get

$$A' \cos A \cos \phi - v \sin A \cos A \sin \phi - 2wv \cos A \sin \phi \cos \phi = 0$$

Formula IV does not correspond to any well-known path, but is presumably the path that would be followed if no constraint existed.

4. Application to correction of observed altitudes. The acceleration is in bearing $270° + A$ and the deflection z of the apparent zenith will be given numerically by

$$\tan z = f/g$$

(Strictly g is not the acceleration due to gravity but is decreased by the 'upwards' acceleration; this is too small to make any practical difference.)

The apparent zenith will be deflected by an amount z in the same direction as the aircraft's acceleration, i.e. in bearing $270° + A$. It can then be easily seen that observed zenith distances are too large when the body observed is in bearing $90° + A$, i.e. observed altitudes are too small on the starboard. This immediately gives the sign with which the correction is to be applied to the observed altitude and leads to the rule that all position lines are to be translated a distance z in a direction $90° + A$, i.e. to the *right* of the track. (In southern latitudes the principal term of z becomes negative because the latitude—and hence $\sin \phi$—is considered as negative; thus the coriolis shift is to the *left* of the track in southern latitudes.)

5. Applications to particular paths. For any method of steering an aircraft it should be possible to determine the quantity A', which determines the track of the aircraft, and thus to deduce the appropriate correction. For instance:

(i) Manual or automatic control, steering a constant course by magnetic or gyro-magnetic compass, corrected for variation.

The track is a rhumb-line, $A' = 0$ and formula II should be used for f.

(ii) Manual or automatic control, steering a constant course by (gyro) Direction Indicator, not compensated for latitude. In this case the aircraft is flying in a fixed direction in space (apart from being constrained to fly at a constant height) and the circumstances can be simulated by flying in the direction of a fixed star, assumed to be on the horizon.

It is not difficult to show that the variation, A', of the azimuth of such a star is

$$A' = \sin \phi (w + v \sin A \sec \phi)$$

and when this is substituted in I,

$$f = awv \sin \phi$$

(iii) As (ii) but with an imposed precession of rate $w \sin \phi$ to counteract the Earth's rotation.

The value of A' obtained is $v \sin A \tan \phi$ since the precession only allows for that part of the variation due to the rotation of the Earth; the resulting value of f is

$$f = 2awv \sin \phi$$

which is the value associated with a great circle. In fact the path flown will be a great circle, except for the variation of latitude with which the rate of precession will cease to correspond.

The course on which there is no displacement of the zenith is now seen to be the course flown by steering by a DI corrected for precession *with the wrong sign.*

(iv) As (iii) but with an imposed precession of rate $w \sin \phi_0$ where ϕ_0 is a latitude different from the latitude of the aircraft.

Clearly

$$f = awv \sin \phi + awv \sin \phi_0$$

which corresponds roughly to the great circle acceleration for the mean of the two latitudes.

6. Discussion. Although the above treatment obscures the fundamental difference between the first term in f (depending on the Earth's rotation) and the remaining terms (independent of the Earth's rotation), it leads very simply to a general formula covering all paths. The application of this general formula is, however, not so simple as anticipated in Section 5 above. In fact, the *Air Almanac* for 1942 and 1943, while tabulating corrections due to the first and the third terms of (I), implicitly ignored the second term when stating that 'if the pilot is flying on a gyro magnetic compass (i.e. flying a rhumb line course) there will be NO Gyro Precession'. The magnitude of the second term depends upon the course, as well as upon speed and latitude, and is not easy to include in a small correction which is already difficult to appreciate and to apply. Strictly, of course, an additional correction should be applied to the coriolis correction (based on the great circle value of the acceleration (III)) as tabulated in the almanacs, depending upon the total deviation of the aircraft's course from a great circle during the period of observation. It is therefore of interest to examine the magnitudes of these deviations, and the resulting corrections that can occur in practice.

Using mean values for the radius of the Earth and for the acceleration due to gravity the deviation of the zenith corresponding to the three terms of (I) is:

$$z = 2'{\cdot}62 V \sin \phi + 0'{\cdot}146 V^2 \sin A \tan \phi - 5'{\cdot}25 VA' \qquad \text{V}$$

where V is the speed of the aircraft in units of 100 knots and A' is measured in degrees per minute. It is immediately seen that a small value of A', of the order of a few tenths only of a degree a minute, will give rise to a large contribution to z. The second term, due to the difference between a great circle and a rhumb line, corresponds to a value of A' of

$$-\frac{1}{36}V \sin A \tan \phi$$

which, in these latitudes and at speeds of the order of 300 knots, is of the order of $0^\circ\cdot1$ per minute only. In two minutes therefore the difference in course between the rhumb line and the great circle will be only $0^\circ\cdot2$. In general, the mean value of the last term of (v) over a period of two minutes will be:

$$-2'\cdot62 V \times \Delta A$$

where ΔA is the total change in degrees in the bearing of the aircraft's track during the observation. A change of 1° at a speed of 300 knots will thus give rise to a deviation of the zenith of $8'$.

Now, the discrimination of most compasses or directional gyros is little, if any, greater than a degree; they can neither be read nor used for automatic control to any appreciably greater accuracy. The possible error due to lack of compass discrimination is therefore five times as great as that due to the difference between great circle and rhumb line courses!

Furthermore, a magnetic compass only defines a rhumb line track if the change of magnetic variation is applied continuously, or, to the accuracy desired, at least more frequently than once every two minutes. In two minutes, at a speed of 300 knots over England, an aircraft may easily experience a change of $0^\circ\cdot15$ in the variation, comparable to the difference between great circle and rhumb line; there are many places on the Earth where the change of variation is much greater. In point of fact a magnetic rhumb line in N.W. Europe approximates very closely to a great circle, though in N. America it is of greater curvature than a true rhumb line.

It is seen that the application of any correction additional to that given by the first term of (v) is not only extremely difficult, but of doubtful value, owing to the lack of compass discrimination. However, it may be argued that statistically better results will be obtained by the application of the second term in (v) even if individual observations are still largely in error. This argument assumes that the average track flown (apart from wind effect) is a rhumb line; this is by no means the case and it would be quite wrong to include the second term generally. It will suffice to point out that in high latitudes it is inconceivable that a pilot will fly a rhumb line, even if the directive force on the magnetic compass were adequate to give him a magnetic course; moreover the rate of change of the magnetic variation is generally high near the poles.

The mean path actually flown by an aircraft over a short period of time while under various types of control is beyond the scope of this article; the theoretical paths have been used in Section 5, but practical considerations may, as has been shown, have an overwhelming influence. Within the limits of sensitivity of the applied control, whether manual or automatic, the path of the aircraft must depend upon personal or mechanical idiosyncrasies superposed upon general dynamical considerations.

Whether it is worth while to apply corrections depending upon the actual track flown by the aircraft as indicated by instruments must depend on circumstances. There would seem to be a case for it when flying by directional gyro in high latitudes if the gyro wander is large and well determined; in fact such a procedure is now used by navigators in the Royal Canadian Air Force. Generally, however, it can be said that the doubtful increase in accuracy is not worth the additional complication—and chance of error—involved.

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