Measuring Arclengths by Stereographic Projection

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Properties of Stereographic Projection

Under stereographic projection, points on the surface of a sphere are mapped to points on a plane as shown in Figure 1. Circles on the sphere map to circles or straight lines on the plane. All straight lines on the plane represent circles in the sphere passing through the north pole projection point. Straight lines passing through the origin or south pole represent great circles and those that do not represent small circles.

Stereographic projection is angle-preserving or conformal as was shown in a published proof in 1695 by none other than Edmund Halley.

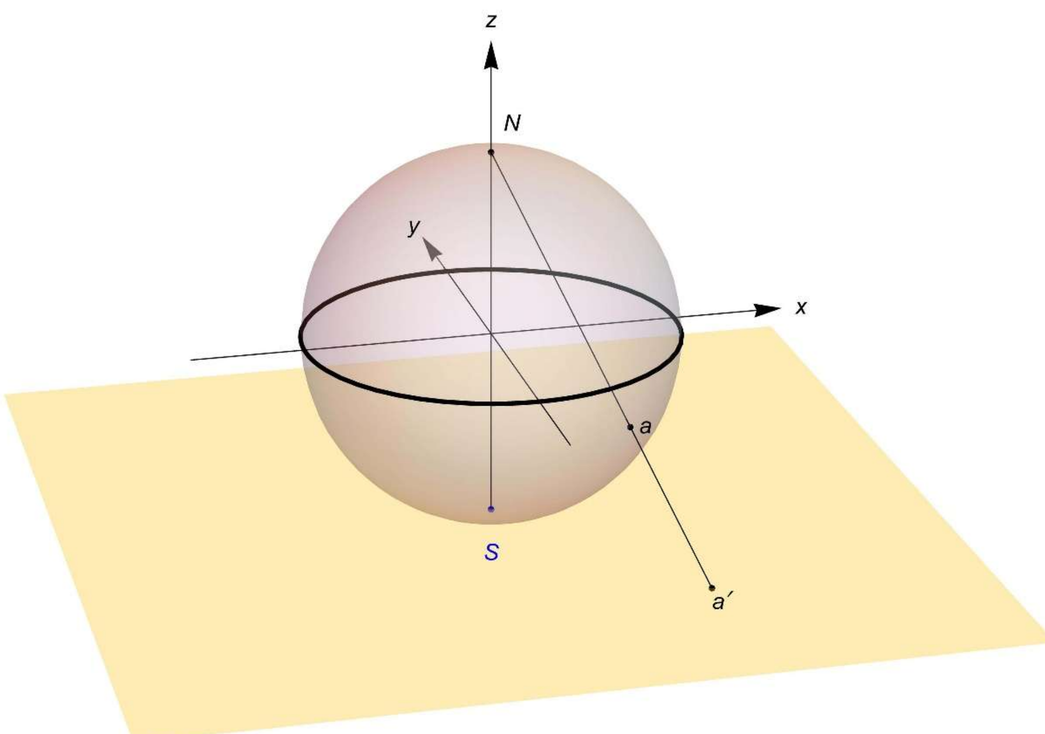

Figure 1: Stereographic projection of the point a on the surface of a sphere of unit radius from the north pole, N, onto the point a' on the plane tangent to the sphere at the south pole, S.

Let the sphere centred at the origin, $0 \equiv (0, 0, 0)$ have unit radius, $r = 1$. Points on the surface of the sphere are identified by their 3D cartesian coordinates (x, y, z) or spherical coordinates corresponding to latitude, L and longitude, λ, where

$$\left(x, y, z\right) = \left(\cos L \cos \lambda, \cos L \sin \lambda, \sin L\right)$$

The zero longitude is in the direction of the x-axis. Stereographic projection will be performed from the north pole $N \equiv (0, 0, 1)$ onto the plane tangent at the south pole $S \equiv (0, 0, -1)$.

Under stereographic projection a point on the surface of the sphere (x, y, z) is mapped to the point $\left(\frac{2x}{1-z}, \frac{2y}{1-z}, -1\right)$. Since the z-coordinate is constant the 3rd component will generally be dropped and the projected point represented as a 2D vector.

The stereographic projection of the equator is a circle of radius $r = 2$ known as the *primitive circle*.

Points that have undergone a rotation away from their original position on the sphere will be denoted by the angle subscript $\angle$ (e.g. $a_\angle$). Points on the plane obtained by stereographic projection are indicated with a prime $'$ (e.g. a' or $a'_\angle$).

Great Circle Distance between Points

For the special case of two points a and b lying on the equator, the great circle distance can be obtained by measuring the angle $\angle a'0b'$ where a' and b' are the stereographic projections of a and b onto the plane. In what follows it will be shown how to obtain the great circle distance when points have been rotated away from the equator.

Let the point a lie on the equator at longitude $-90°+\lambda$. Its cartesian coordinates are $\left(\sin \lambda, -\cos \lambda, 0\right)$ and b be the point $\left(0, -1, 0\right)$. Rotate the point a and pole S by an angle ΔL about the x-axis. This accomplished by multiplying their cartesian coordinates by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Delta L & -\sin \Delta L \\ 0 & \sin \Delta L & \cos \Delta L \end{pmatrix}$$

Label the resulting points $a_\angle$ and $S_\angle$. Under stereographic projection the points a, $a_\angle$ and $S_\angle$ map to

$$a: \rightarrow a' = 2\left(\sin \lambda, -\cos \lambda\right)$$

$$a_\angle :\rightarrow a'_\angle = \frac{2}{1+\cos \lambda \sin \Delta L}\left(\sin \lambda, -\cos \Delta L \cos \lambda\right)$$

$$S_\angle :\rightarrow S'_\angle = \left(0, \frac{2\sin \Delta L}{1+\cos \Delta L}\right)$$

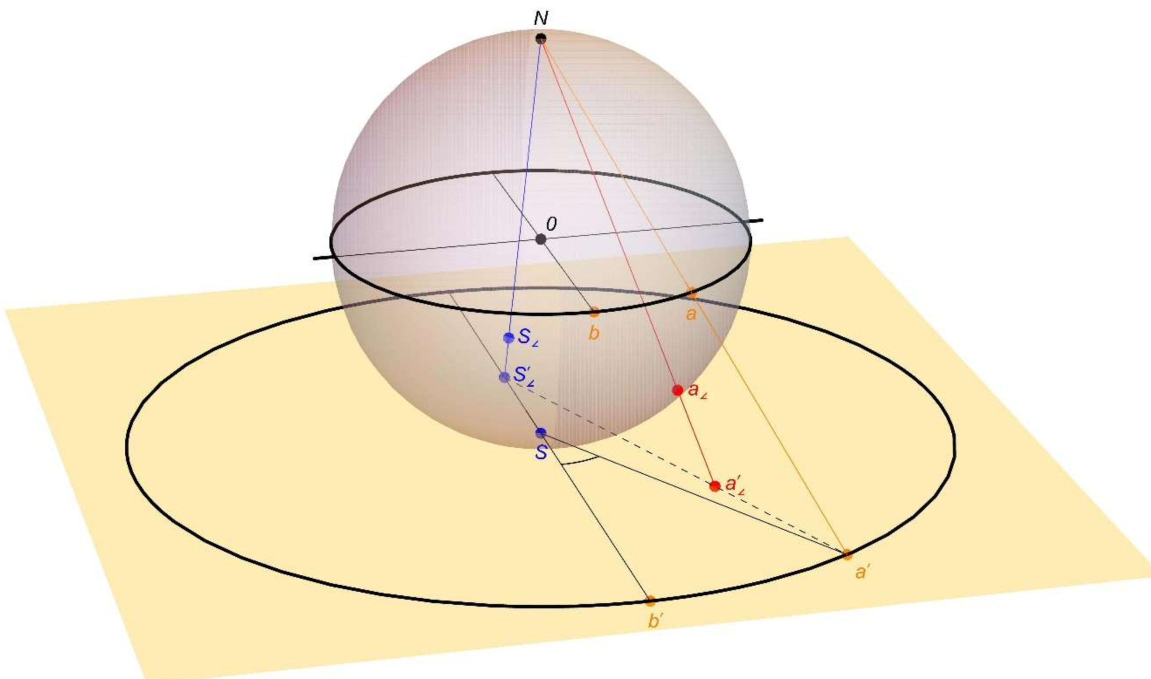

Figure 2: 3D representation of the construction for finding the great circle distance between 2 points. The points a and b sit on the equator with $b = (\, 0, -1, 0 \,)$. Upon rotation about the x-axis, the point a maps to $a_\angle$ and the south pole, S, maps to $S_\angle$. The images of these rotated points under stereographic projection are $a'_\angle$ and $S'_\angle$ respectively.

These three points are colinear which can be demonstrated by showing that the determinant

$$\begin{vmatrix} \sin\lambda & -\cos\lambda & 1 \\ \dfrac{\sin\lambda}{1+\cos\lambda\sin\Delta L} & \dfrac{-\cos\Delta L\cos\lambda}{1+\cos\lambda\sin\Delta L} & 1 \\ 0 & \dfrac{\sin\Delta L}{1+\cos\Delta L} & 1 \end{vmatrix} = 0 .$$

This means that if $a_\angle$ is a point on the sphere lying on the great circle with pole $S_\angle$ then the image a' under stereographic projection of the point a can be found by extending a straight line from the pole $S'_\angle$ through the point $a'_\angle$ until it intersects the primitive circle. This procedure is called *reducing* $a'_\angle$ *to the primitive circle*. The great circle distance between a and b is then obtained by measuring the angle $\angle a'0b'$.

The process described above straightforwardly generalizes for finding the great circle distance between arbitrary pairs of points.

Since the points $S'_\angle$, $a'_\angle$ and a' lie in a straight line on the projection plane, the points $S_\angle$, $a_\angle$ and a on the sphere lie on a small circle passing through the north pole, N. The points $S_\angle$, $a_\angle$, a and N are therefore coplanar in 3 dimensions.

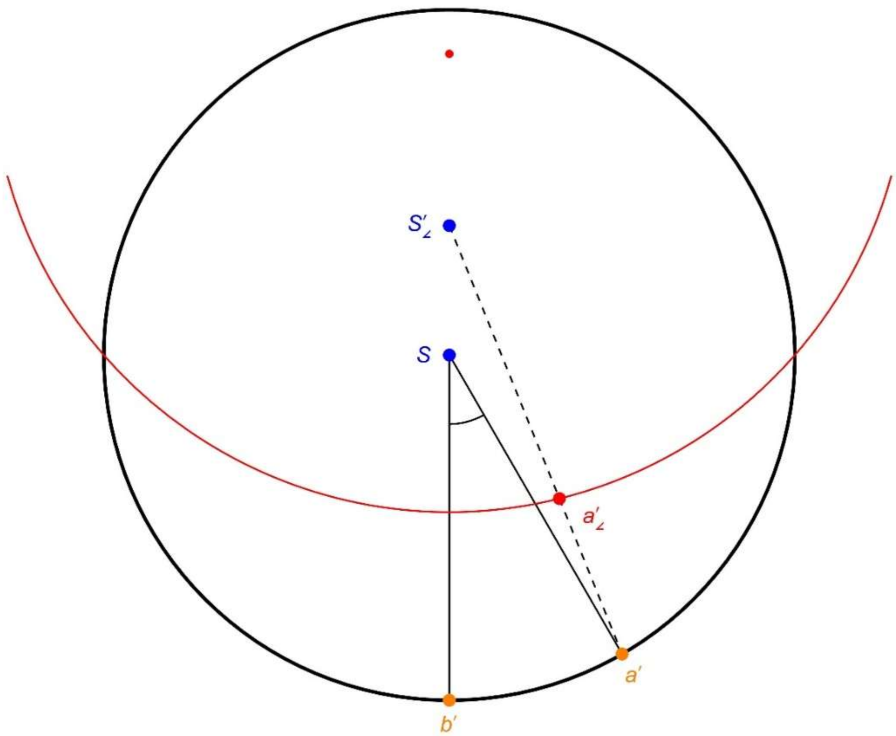

Figure 3: 2D view of the stereographic projection plane in Figure 2 showing the construction used to find the point a'. The great circle distance from a to b is found by measuring $\angle a' Sb'$. Also shown in red is stereographic projection of the great circle on the sphere containing $a_\angle$ with pole at $S_\angle$.

Small Circle

Let two points a and b lie on a small circle with a pole at S. The angle subtended by the arc of the small circle $\frown ab$ at its centre can be found by measuring the angle $\angle\overline{a}'Sb'$ where b' is the stereographic projections of b onto the plane and $\overline{a}'$ is a point lying along the ray Sa'. The point a' is stereographic projections of a onto the plane.

Let the point a lie at longitude $-90°+\lambda$ and latitude $-L$. Its cartesian coordinates are $(\cos L \sin\lambda, -\cos L\cos\lambda, -\sin L)$. As before rotate the point a and pole S by an angle ΔL about the x-axis to obtain the points $a_\angle$ and $S_\angle$ respectively. Under stereographic projection these points map to

$$a_{\angle} :\to a'_{\angle} = \frac{2}{1+\sin L\cos\Delta L+\cos L\cos\lambda\sin\Delta L}\left(\cos L\sin\lambda, \sin L\sin\Delta L-\cos L\cos\lambda\cos\Delta L\right)$$

$$S_{\angle} :\to S'_{\angle} = \left(0, \frac{2\sin\Delta L}{1+\cos\Delta L}\right)$$

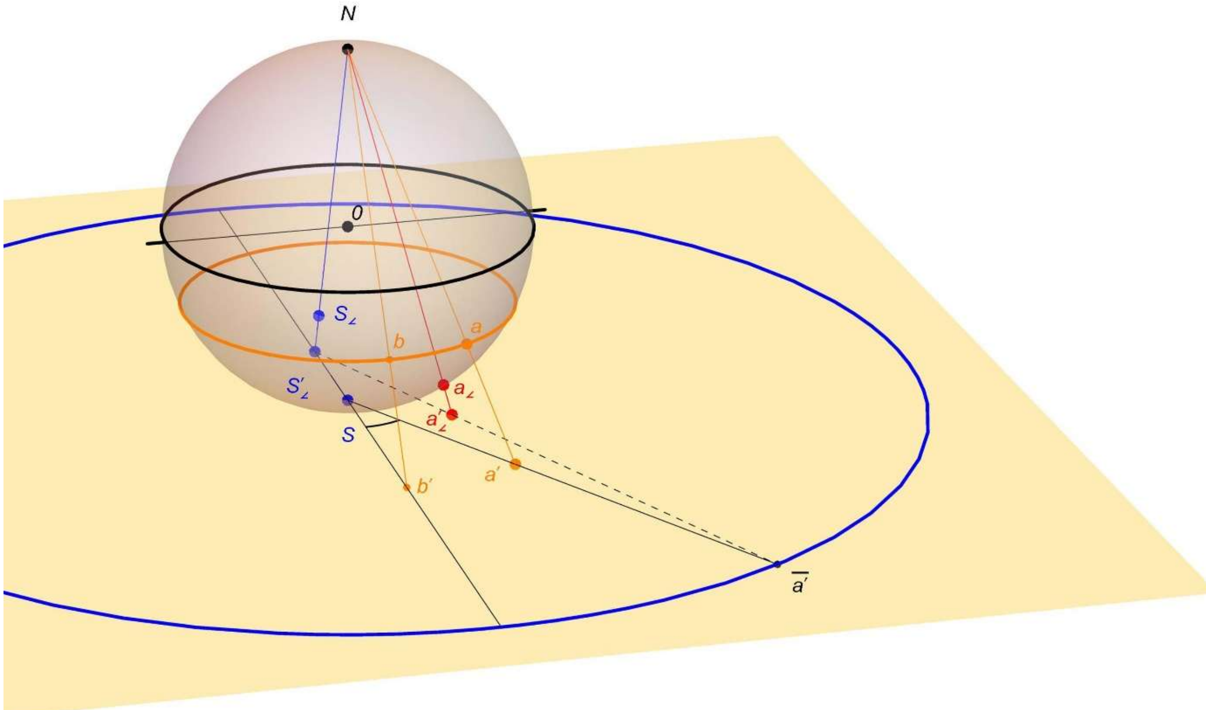

Figure 4: 3D representation of the construction for finding the angle subtended at the centre of a small circle by an arc between points *a* and *b* on its circumference. The point *b* lies in the *y*-*z* plane. The small circle is shown in orange with its pole at the south pole, *S*. As in Figure 2, upon rotation about the *x*-axis, the point *a* maps to $a_{\angle}$ and the south pole, *S*, maps to $S_{\angle}$. The images of these rotated points under stereographic projection are $a'_{\angle}$ and $S'_{\angle}$ respectively. The radius of the blue circle depends only on the radius of the small circle.

The aim now is to use these two points to locate a point $\overline{a'}$ lying along the ray Sa' and hence allow the required angle to be measured directly on the plane. Points on Sa' take the form $\overline{a'} = \mu\left(\sin\lambda, -\cos\lambda\right)$ for some value of the factor μ. The intersection of the ray Sa' and the line passing through $S'_{\angle}$ and $a'_{\angle}$ can be found by again applying the collinearity condition by requiring that the determinant

$$\begin{vmatrix} \mu \sin\lambda & \mu \sin\lambda & 1 \\ \dfrac{2\cos L \sin\lambda}{1+\sin L\cos\Delta L+\cos L\cos\lambda\sin\Delta L} & 2\dfrac{\sin L\sin\Delta L-\cos L\cos\lambda\cos\Delta L}{1+\sin L\cos\Delta L+\cos L\cos\lambda\sin\Delta L} & 1 \\ 0 & \dfrac{2\sin\Delta L}{1+\cos\Delta L} & 1 \end{vmatrix} = 0$$

After some algebra it is found that the determinant vanishes when

$$\mu = \frac{2\cos L}{1-\sin L} = 2\tan\left(\frac{\pi}{4}+\frac{L}{2}\right) = 2\tan\left(\frac{\pi - r}{2}\right)$$

where r is the radius of the small circle. The parameter μ is the radius of the blue circle.

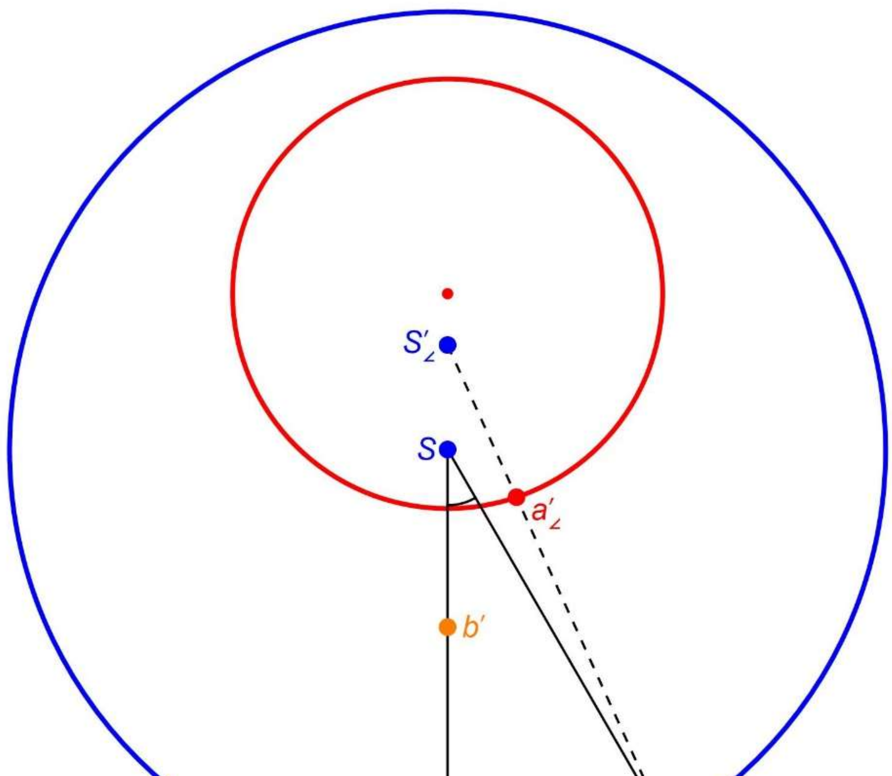

Figure 5: 2D view the stereographic projection plane in Figure 4 showing the construction used to find the point $\overline{a'}$ The angle subtended at the centre by the $\frown ab$ is found by measuring $\angle \overline{a'} S b'$. Also shown in red is the stereographic projection of the small circle on the sphere containing $a_{\angle}$ with pole at $S_{\angle}$.