A METHOD OF CALCULATING THE EXACT POINT OF INTERSECTION OF "SUMNER LINES" IN NAVIGATION

By Clarence E. Woodman

THE usual way of working a "Sumner" observation is to take an altitude of the sun, and, assuming two latitudes—one above and one below the supposed "dead-reckoning" latitude—to calculate a separate longitude for each assumed latitude, and to connect the points so found by a straight line on the chart. The ship must be somewhere on that line. Then, an hour or two afterwards, another altitude is taken, and two new longitudes are calculated, using the same assumed latitudes as before (unless the run of the ship has made too much northing or southing: in which case another pair of latitudes may be chosen.) These second longitude-points are likewise marked on the chart, and connected by a straight line. The ship must also be somewhere on this second line. Then, if necessary, the first line is brought up to the second, by allowing for and plotting down the course and distance run; and the intersection of the two lines must mark the ship's place at the second observation.

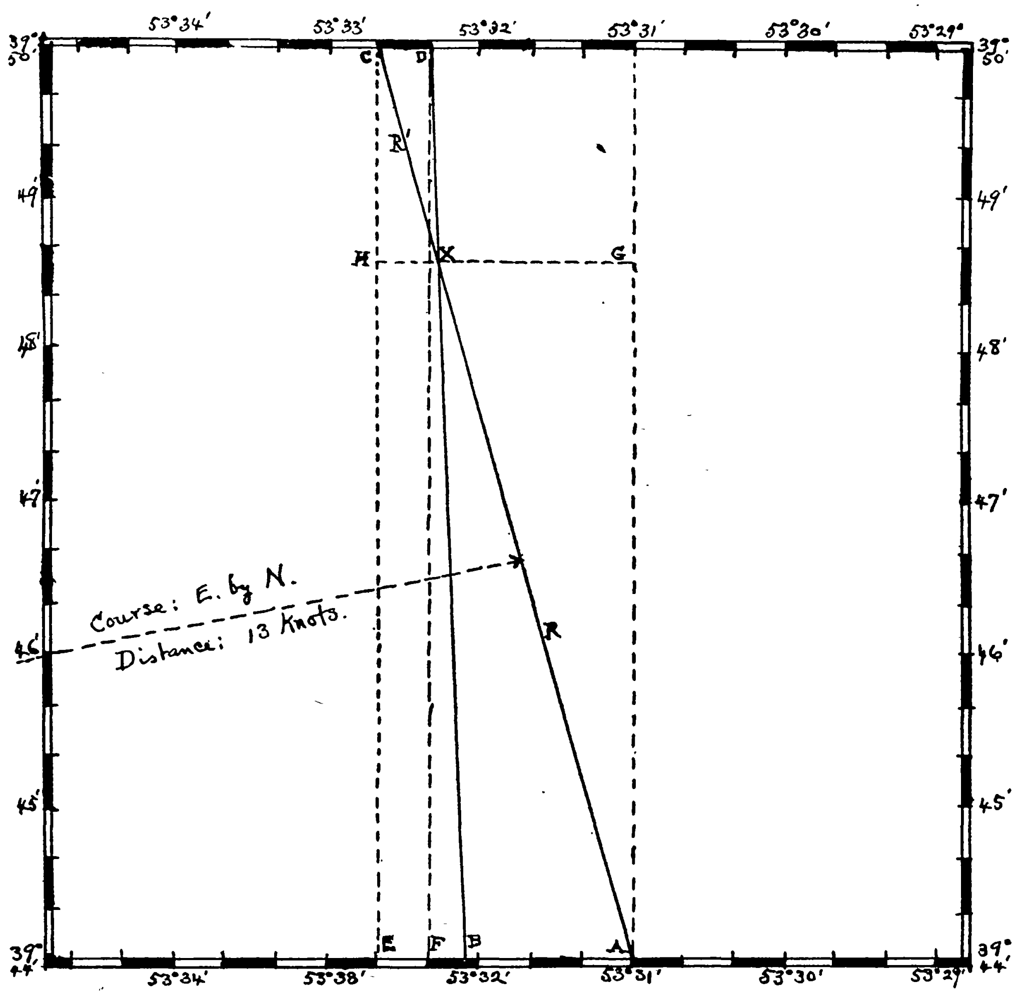
When the angle at the intersection of the two lines is over 45°, and the chart is on a large scale, and the ruler has a straight edge and the pencil a fine point, this method does very well. The resultant latitude and longitude will probably be correct within a minute or so. But when the angle is very acute the "fix" is not so good: the lines coalesce for some appreciable distance, and it is hard to tell just where the intersection really is. The "pencil-and-chart" method is a rough one at best; and so the following is suggested as a method whereby the intersection-point can be calculated (to the nearest second, if required) and all uncertainty avoided as to the precise "fix."

The accompanying illustration shows two Sumner lines, *AC* and *BD*, drawn on the chart: the assumed latitude being 39° 44′ and 39° 50′, and the resulting longitudes being in the first case

53° 30′ 59″ and 53° 32′ 39″, and in the second 53° 32′ 07″ and 53° 32′ 20″. The "fix" is somewhere around X—or about latitude 39° 48′ 40″ and longitude 53° 32′ 16″. To calculate this point exactly:

In the triangle ABX the angle A (or CAE) may be found by

$$\tan A = \frac{CE}{AE}.$$



But CE is the difference between the assumed latitudes, and AE is the difference between the first longitude at the first latitude and the first longitude at the second latitude. Again the angle B may be found by

$$\tan (180° - B) = -\tan B = \frac{DF}{BF}.$$

But $DF = CE$, the latitude-difference as before, and BF is the difference between the second longitude at the first latitude and the second longitude at the second latitude. Having found A and B, we get the angle at X by

$$X = 180° - (A + B).$$

This angle X, can be found at once, if the *azimuth* of the sun is observed at (or calculated for) the moment of each altitude-observation,—as the ordinary sea-practice; for this angle is, of course, the difference between the azimuths.

Now in the triangle ABX we get the side R by

$$R = \frac{AB \sin B}{\sin X}.$$

AB is known: as it is the difference between the first and the second longitudes at the first latitude. Having found R, we get AG, the *plus* latitude-correction, by

$$AG = R \sin A$$

and GX, the *plus* longitude correction, by

$$GX = R \cos A.$$

We can check these results by using the upper triangle, CDX (C, of course, being equal to A, and D to B), and getting R' by

$$R' = \frac{CD \sin B}{\sin X}$$

(CD is known: being the difference between the first and the second longitude at the second latitude.) So we get CH, the *minus* latitude-correction, by

$$CH = R' \sin A$$

and HX, the *minus* longitude-correction, by

$$HX = R' \cos A.$$

We have, therefore, the following rules:*

1. Divide the difference between the two assumed latitudes by the difference between the longitudes resulting from the *first* observation. The quotient is the tangent of an angle, which call A.

2. Divide the same latitude-difference by the difference between the longitudes resulting from the *second* observation. The

*Reduce all arcs to seconds. If the ship has moved in the interval, bring the first observation-line up to the place of the second, by Traverse Table in the usual manner.

quotient is the tangent of an angle, which call 180° − *B* (which is the same as *minus* tan *B*).

3. Subtract the sum of these angles, *A* and *B*, from 180°. The remainder is a third angle, which call *X*.

4. Multiply the sine of *B* by the difference between the longitudes resulting from the first and second observations at the *first* assumed latitude, and divide the product by the sine of *X*. Call the result *R*.

5. Then the true latitude is the least assumed latitude *plus* *R* sin *A*; and the true longitude is the least longitude at the first observation *plus* *R* cos *A*.

6. For a check: Multiply the sine of *B* by the difference between the first and the second longitudes at the *second* assumed latitude, and divide the product by the sine of *X*. Call the result *R′*. Then the true latitude is the greatest assumed latitude *minus* *R′* sin *A*, and the true longitude is the greatest first-observation longitude *minus* *R′* cos *A*.

EXAMPLE

At sea, at about half past three p.m., ship's time, the latitude by dead reckoning was 39° 46′ N. and the approximate longitude 53° 45′ W.; barometer, 29.80; thermometer, 82°; height of eye above sea, 26 ft.; index-error of sextant, −10″. The Greenwich mean time by chronometer was 7h 03m 34s.

Observed alt. sun's lower limb =	40° 49′ 00″	Index	= −00′ 10″
Correction = +	9′ 43″	Dip	= −05′ 00″
		Refraction	= −01′ 02″
		Semidiameter	= +15′ 48″
		Parallax	= +00′ 07″
True altitude sun's centre	= 40° 58′ 43″	Correction	= +09′ 43″
Sun's declination Greenwich noon =	+17° 25′ 21″.9	Hourly diff.	= 39″.58
		Elapsed time	= 7
Correction	= − 4′ 37″.1		
		Correction	= 277″.06
			= 4′37″1
True declination =	+17° 20′ 45″		
Polar distance (P) =	72° 39′ 15″		
Equation of time Greenwich noon =	+5m 58s.85	Hourly diff.	= −0s.206
Correction = −	1s.44	Elapsed time	= 7
True equation of time	= +5m 57s.4	Correction	= −1s.442

To calculate the hour-angles (t) the ordinary "sine-formula" is used; viz:

$$\sin\tfrac{1}{2}t = \sqrt{\frac{\cos s \sin(s-\text{alt.})}{\cos \text{lat.} \sin p}}$$

(where s is the half-sum of the altitude, latitude and polar distance).

1st assumed latitude = 39° 44′

alt = 40° 58′ 43″		
lat = 39° 44′ 00″	sec	= 0.11406
p = 72° 39′ 15″	cosec	= 0.02021
2 | 153° 21′ 58″		
s = 76° 40′ 59″	cos	= 9.36237
s − alt = 35° 42′ 16″	sin	= 9.76612
	2 |	9.26276
$\sin\frac{1}{2}t$		= 9.63138
Local app. time		= 3h 22m 41s.6
Equation of time		= + 5m 57s.4
Local mean time		= 3h 28m 39s.0
Green. mean time		= 7h 03m 34s.0
Longitude in time		= 3h 34m 55s.0
Longitude in arc		= 53° 43′ 45″

2d assumed latitude = 39° 50′

alt = 40° 58′ 43″		
lat = 39° 50′ 00″	sec =	0.11469
p = 72° 39′ 15″	cosec =	0.02021
2 | 153° 27′ 58″		
s = 76° 43′ 59″	cos =	9.36076
s − alt = 35° 45′ 16″	sin =	9.76665
	2 |	9.26231
	$\sin\frac{1}{2}t$ =	9.63116
Local app. time		= 3h22 m34s.9
Equation of time		= + 5m 57s.4
Local mean time		= 3h 28m 32s.3
Green. mean time		= 7h 03m 34s.
Longitude in time		= 3h35m01s.7′
Longitude in arc		= 53° 45′ 26′

After the ship had run 13 knots, on a course E. by N., the first observed longitudes were brought up to the ship's place by the Traverse Table; giving for that course and distance a diff. lat. of 2′ 32″, and a diff. long. of 12′ 45″. This diff. lat. brings the dead reckoning latitude up to 39° 48′ 32″, and this diff. long. applied to the previously-found longitudes gives: for assumed latitude 39° 44′ a longitude of 53° 31′ 00″, and for assumed latitude 39° 50′ a longitude of 53° 32′ 41″. Thedifference between these longitudes is 101″.

Another sight was then taken, in d.r. lat. 39° 48′ 32″ and d.r long. 53° 32′ 15″, and the new altitude and polar distance used with the two assumed latitudes as before; the Gr. M.T. by chronometer being 8h 02m 12s.

Observed altitude lower limb	= 29° 29′ 20″	Index	= −00′ 10″
Correction	= + 9′ 10″	Dip	= −05′ 00″
		Refraction	= −01′ 36″
		Sedidiameter	= +15′ 48″
		Parallax	= +00′ 08″
True altitude sun's centre	= 29° 38′ 30″	Correction	= +09′ 10″
Sun's declination, Green. noon	= +17° 25′ 21″.9	Hourly diff.	= 39″.6
Correction	= − 5′ 16″.8	Elapsed time	= 8

True declination	= +17° 20′ 05″	Correction =	316″.8
Polar distance (p)	= 72° 39′ 55″	=	5′ 16″.8
Equation of time Green. noon	= +5m 58s.85	Hourly diff. =	− 0s.206
Correction	= − 1s.65	Elapsed time =	8
True equation of time	= +5m 57s.2	Correction =	− 1s.648

1st assumes latitude = 39°44′		2d assumed latitude = 39° 50′	
alt = 29° 38′ 30″		alt = 29° 38′ 30″	
lat = 39° 44′ 00″ sec =	0.11406	lat = 39′° 50 00″ sec =	0.11469
p = 72° 39′ 55″cosec =	0.02019	p = 72° 39′ 55″cosec =	0.02019
2 | 142° 02′ 25″		2 | 142° 08′ 25″	
s = 71° 01′ 12″.5 cos =	9,51220	s = 71°04′12″.5 cos =	9.51109
s − alt = 41°22′42″.5 sin =	9.82022	s − alt = 41°25′42″.5 sin =	9.82065
2 |	9.46667	2 |	9.46662
$\sin\frac{1}{2}t$ =	9.73333	$\sin\frac{1}{2}t$ =	9.73331
Local app. time	= 4h22m06s.4	Local app. time	= 4h 22m05s.5
Equation of time	= + 5m57s.2	Equation of time	= + 5m57s.2
Local mean time	= 4h 28m 03s.6	Local mean time	= 4h 28m 02s.7
Green. mean time	= 8h 02m 12	Green. meantime	= 8h02m 12s
Long. in time	= 3h 34m 08s.4	Long, in time	= 3h 34m09s.3
Long. in arc	= 53° 32′ 06″	Long. in arc	= 53° 32′ 20,

The difference between these longitudes is 14″.

Referring to the chart-diagram, we tabulate the necessary factors as follows:

AE = (53° 32′ 41″ − 53° 31 00″) = 101″
BF = (53° 32′ 20″ − 53° 32′ 06″) = 14″
AB = (53° 32′ 06″ − 53° 31′ 00″) = 66″
CD = (53° 32′ 41″ − 53° 32′ 20″) = 21″
CE = (39° 50′ 00″ − 39° 44′ 00″) = 360″

In triangle AEC,

$$\tan A = \frac{CE}{AE} = \frac{360}{101} \quad \begin{matrix}\log = 2{,}55630 \\ \log = 2.00432\end{matrix}$$

log tan A = 0.55198
A = 74° 19′ 41″.8

$$-\tan B = \frac{DF}{BF} = \frac{360}{14} \quad \begin{matrix}\log = 2.55630 \\ \log = 1.14613\end{matrix}$$

log − tan B = 1.41017n
B = 92° 13′ 37″.4

$A+B=166°\ 33'\ 19''.2$

$X=180°-(A+B)=13°\ 26'\ 40''.8$

$$R=\frac{AB \sin B}{\sin X}=\frac{66 \sin 92°\ 13'\ 37''.4}{\sin 13°\ 26'\ 40''.8}$$

log 66	=1.81954
log sin B	=9.99967
log cosec X	=0.63358
log R	=2.45279. R=283.65

In triangle AGX,

$AG=R \sin A$, and $GX=R \cos A$.

log R	=2.45279		=2.45279
log sin A	=9.98355	log cos A	=9.43157
log AG	=2.43634	log GX	=1.88436
AG	=273''.1=4' 33''.1	GX	=76''.6=1' 16''.6
(the *plus* lat. correction)		(the *plus* long. correction)	

The true latitude, therefore, is (39° 44'+4' 33''.1)=39° 48' 33''.1, and the true longitude is (53° 31'=1' 16''.6)=53° 32' 16''.6.

CHECK

In triangle CDX,

$$R'=\frac{CD \sin B}{\sin X}=\frac{21 \sin 92°\ 13'\ 37''.4}{\sin 13°\ 26'\ 40''.8}$$

log 21	=1.32222
log sin B	=9.99967
log cosec X	=0.63357
log R'	=1.95546 R'=90.25

In triangle CHX,

$CH=R' \sin A$, and $HX=R' \cos A$.

log R'	=1.95546		=1.95546
log sim A	=9.98355	log cos A	=9.43157
log CH	=1.93901	log HX	=1.38703
CH	=86''.9=1' 26''.9	HX	+24''.4
(the *minus* lat. correction)		(the *minus* long. correction)	

These corrections are *subtractive* from the highest latitude and the greatest longitude respectively; and this operation gives thesame result as before: *i.e.*, true latitude=(39° 50' −1' 26''.9)=39° 48' 33''.1, and true longitude=(53° 32' 41''−24''.4)=53° 32' 16''.6.

This method is evidently well adapted for land observation, with transit or with sextant and artificial horizon. The approximate map-position is used instead of "dead reckoning"; and, as the observations are made in the same place, no allowance for "run of ships" is necessary. The degree of precision arrived at is limited only by the variant and inevitable inaccuracies of observation.

NEWMAN HALL, UNIVERSITY CATHOLIC CHAPEL,
BERKELEY, CALIFORNIA.