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The "Equation of Time" Made Simple?

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Introduction

Historically the time of day was anchored by noon, or the time at which the sun reached its highest point in the sky. However, with the advent of clocks that were accurate enough to keep time over weeks and months a problem arose. Clocks are made to keep "mean" time, namely they divide the day into hours, and the hour is defined to be $1/24^{th}$ of the length of a day averaged over a year. For such a clock noon as given by mean time does not always coincide with noon as defined by the sun. In fact the two are coincident only four times a year.

The difference in time between solar noon as displayed by a sundial and noon mean time as kept by a clock is given by the "Equation of Time" (EoT). More generally the EoT gives the difference at any time between solar time and mean time. The major contributors to this difference are the "eccentricity" or elliptical nature of the earth's orbit around the sun, and the tilt of the earth's axis relative to its orbital plane. The values for the EoT are usually given in a table with one value per day, namely the difference between solar time and mean time at noon at the Greenwich Observatory. Because the year is approximately 365.25 days long this table changes every year in a four-year cycle, and reverts to the original values (approximately) every leap year.

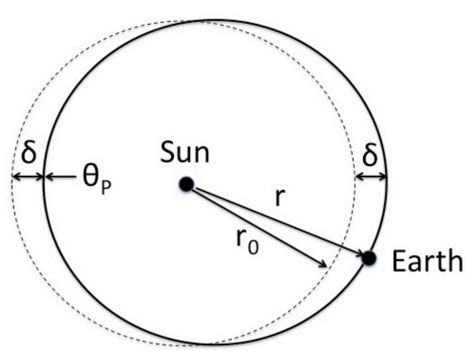
The goal of this note is to develop a relatively simple expression for the EoT using a few basic relationships from geometry and physics. The starting place is the realization that the EoT would be zero for all times if the earth's orbit was circular and its rotation axis was perpendicular to its orbital plane. The second thing to realize is that the two effects giving rise to non-zero EoT values can be treated independently and their effects can be added together.

Effect of Elliptical Orbit

First we'll consider the earth's elliptical orbit. Because this ellipse has a small eccentricity of only 0.017 (defined as $\varepsilon = \sqrt{((a^2-b^2)/a^2)}$ where a is the major axis and b is the minor axis), we can treat the orbit as circular and introduce a simple radial oscillation as a small perturbation:

$$r(\theta) \cong r_0 - \delta \cos(\theta - \theta_P)$$

Here r is the orbit radius at a given orbital angle θ, r_0 is the average radius, δ is half the difference between the obit's radius at aphelion and at perihelion, and θ_P is the orbital angle of the perihelion. Note that because of the inverse square nature of the gravitational force a planetary orbit has exactly one radial oscillation per orbit, namely its orbit is an ellipse with the sun at the focus.



Next we invoke the law of Conservation of Angular Momentum, which for a planet specifies:

$$mrv_C = mr^2\omega = mr_0^2\omega_0 = \text{constant}$$

where m is the mass of the planet, v_C is its circumferential velocity, ω is its orbital angular velocity, and ω_0 is the value of ω at $r=r_0$. Combining this relationship with the equation for $r(\theta)$ we get:

$$\omega(\theta) = \omega_0 r_0^2/r^2 \cong \omega_0/(1 - (\delta/r_0)\cos(\theta-\theta_P))^2$$

The ratio δ/r_0 is equivalent to the eccentricity, so $\delta/r_0 = \varepsilon << 1$ and we can further simplify this as:

$$\omega(\theta) \approx \omega_0 \times (1 + (2\delta/r_0)\cos(\theta-\theta_P)) = \omega_0 + 2\varepsilon\omega_0 \cos(\theta-\theta_P)$$

The second term in the expression for $\omega(\theta)$ gives the amount that the angular velocity deviates from its average value, ω_0, as the earth moves around its orbit. Note that maximum angular velocity occurs when $\theta = \theta_P$ and the earth is closest to the sun. If we integrate $\omega(\theta)$ over time we can get an expression for $\theta(t)$. This integration can be done iteratively by first ignoring the small second term to get $\theta(t)-\theta_P \approx \omega_0 t$, where θ at t=0 is defined to be θ_p. For the next integration iteration this approximation can then be substituted back into the second term yielding:

$$\theta(t)-\theta_P \approx \omega_0 t + 2\varepsilon \sin(\omega_0 t)$$

Now the second term gives the orbital angular deviation of the planet from the location it would have had if its orbit were circular. If the planet is advanced in its orbit by such a deviation, it will have to rotate further around on its axis in order to have the same angular alignment with the sun. In other words for a particular location of the earth's surface, solar noon would be delayed while the earth rotates the additional amount given by the second term. The time delay for this additional rotation is just given by the second term divided by the angular velocity, which we can take as ω_0. Hence the second term divided by ω_0 is just the negative of the Equation of Time, provided that EoT is defined as the value of solar time minus the value of mean time:

$$EoT_E \approx -(2\varepsilon/\omega_0)\sin(\omega_0 t) \approx -(2\varepsilon/\omega_0)\sin(\theta-\theta_P)$$

Here the subscript E signifies that this is the EoT term resulting from the earth's elliptical orbit, and the expression is given either as a function of time or of orbit angular position. EoT_E is plotted for the year 2013 as the blue curve in the figure at the end.

Effect of Rotation Axis Tilt

Next we will develop a formula for the Equation of Time term resulting from the tilt of the earth's axis, which we'll name EoT_T. The starting place, as before, is a circular orbit for the earth with its rotation axis perpendicular to the orbital plane, which results in a zero value for EoT.

Consider the location on the earth's surface where the sun is currently directly overhead, namely at the zenith. We'll call this location the zenith point. The zenith point is located on a meridian, or great circle line extending from the North Pole to the South Pole. For every point along this meridian the solar time is currently noon. Note that this is true regardless of whether the earth's axis is perpendicular or tilted relative to orbital plane.

Now we will tilt the earth by rotating it through an angle α about an axis passing through its equator. Furthermore we associate a direction with the tilt axis such that positive α corresponds to a counter-clockwise (or right-handed) rotation around the axis. Note that as the tilted earth orbits around the sun the tilt axis will continue point in the same direction due to the conservation of angular momentum.

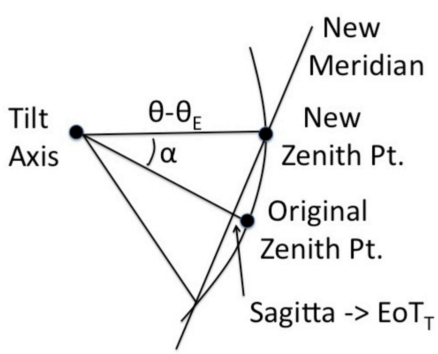
Around the earth's orbit there are four special cases to consider: the tilt axis is parallel or anti-parallel to the earth's direction of motion and the tilt axis is parallel or anti-parallel to the radius vector pointing from the sun to the earth. For positive α the first two cases are just the winter and summer solstices and the latter two cases are the spring and autumn equinoxes. For our discussion we'll define the orbital angular location of the winter solstice to be θ_S.

For the two solstices, when $\theta = \theta_S$ or $\theta_S+\pi$, the zenith point as defined above moves along its own meridian as the earth is tilted. In other words the meridian defining solar noon does not change and thus EoT_T is zero for any value of the tilt angle α.

For the equinoxes, when $\theta = \theta_S+\pi/2$ or $\theta_S+3\pi/2$, the zenith point is actually located on the tilt axis. Therefore neither the zenith point nor its associated meridian move as the earth is tilted, so as with the solstices, EoT_T is zero for all α.

For the more general situation let us consider the range of θ between the spring equinox and the summer solstice. (The result will be valid for all θ, but for visualization purposes it is useful to focus on one quadrant.) As the earth is tilted by an angle α, the zenith point moves counter-clockwise along a circular path on the earth's surface. The center of this circle is just the tilt axis, and its radius is equal to the earth's orbital angular distance from the spring equinox, or $\theta–\theta_S–\pi/2 = \theta–\theta_E$, where θ_E is the orbital angle of the equinox. (As a sanity check note that the radius goes to zero at the equinox meaning the zenith point doesn't move, and it goes to $\pi/2$ at the solstice meaning the zenith point moves along a great circle, namely its meridian.)

As the earth is tilted and the zenith point moves along a circle, the meridian associated with the moving zenith point cuts across the circle forming a chord. Because the circle is centered on the tilt axis, which is on the equator, the meridian chord interests the circle at points that are equidistant from the original location of the zenith point. Furthermore the angular distance of the original zenith point from this new meridian is equivalent to EoT_T. By definition it is currently solar noon along the meridian associated with the new zenith point, but it would have been solar noon at the original zenith point if the earth had not been tilted.



The distance between the original zenith point and the meridian associated with the new zenith point is just the sagitta of the chord formed by new meridian. If we were dealing with planar geometry the sagitta would be given by the radius of the circle times $(1–\cos\alpha)$. However, in this case we need to use spherical geometry for the solution. Using Napier's Rules[1] for right spherical triangles we can write:

$$\text{Sagitta} = \theta–\theta_E – \arctan(\cos\alpha \tan(\theta–\theta_E))$$

[1] Bowditch, Nathaniel; *The American Practical Navigator*; US NIMA; 2002 Edition; p 326.

where $\theta-\theta_E$ is the radius of the circle and α is the half angle subtended by the chord. This is equivalent to EoT_T expressed in units of orbital angle, so we divide by ω_0 to obtain time units:

$$EoT_T = (\theta-\theta_E - \arctan(\cos\alpha \tan(\theta-\theta_E))) / \omega_0$$

The expression above was derived for the half year around the spring equinox, however, if θ_E is taken to mean the orbital angle of either equinox, then the expression can be used for both. EoT_T is plotted for the year 2013 as the green curve in the figure at the end.

Note that near an equinox, for small values of $\theta-\theta_E$, EoT_T simplifies to $(\theta-\theta_E) \times (1-\cos\alpha) / \omega_0$, which is just the result from planar geometry. Also note that the expression for EoT_T is difficult to calculate near a solstice where the arctan argument approaches infinity. However, near either solstice the expression can be simplified to $(\theta-\theta_S) \times (1-1/\cos\alpha) / \omega_0$, where θ_S is the orbital angle of the solstice. From these simplified forms it can be seen that EoT_T is zero at both equinoxes and solstices.

Summary

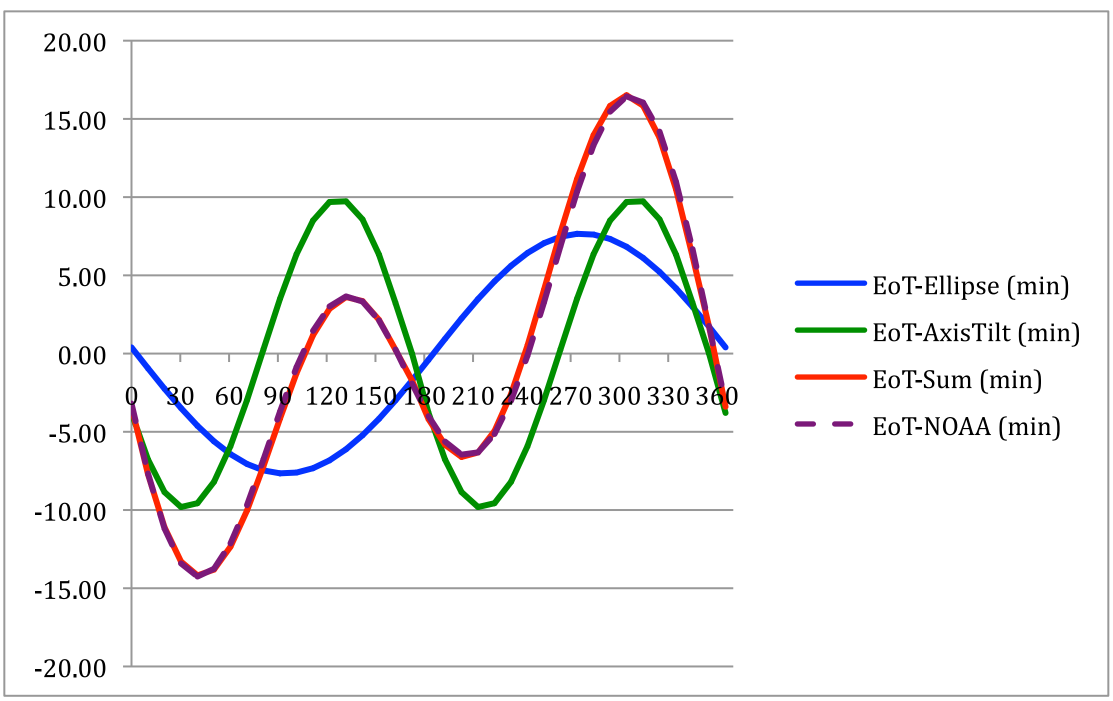
Finally we obtain the complete Equation of Time by adding EoT_E and EoT_T:

$$EoT \approx (-2\varepsilon \sin(\theta-\theta_P) + \theta-\theta_E - \arctan(\cos\alpha \tan(\theta-\theta_E))) / \omega_0$$

Here EoT is expressed as a function of the orbital angle, θ, of the earth. We can instead give EoT as a function of mean time by replacing θ with $\omega_0 t$, θ_P with $\omega_0 t_P$, and θ_E with $\omega_0 t_E$:

$$EoT \approx (-2\varepsilon \sin \omega_0(t-t_P) + \omega_0(t-t_E) - \arctan(\cos\alpha \tan \omega_0(t-t_E))) / \omega_0$$

This expression is shown as a function of calendar date for 2013 by the red curve in the figure below. The only parameters required for the calculation are ε, t_P, t_E, α, and ω_0. For comparison we show a calculation of EoT by NOAA[2] for 2013 as the dashed curve in the figure. As can be seen the expression we have developed for EoT agrees very well with the NOAA calculation.



[2] Taken from NOAA_Solar_Calculations_Day.xls @ www.srrb.noaa.gov/highlights/sunrise.